The Open University

A first level
Interdisciplinary
course

Using *Mathematics*

BLOCK A
MATHEMATICS AND MODELLING

Functions

CHAPTER
A3

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Prepared by the course team

About this course

This course, MST121 *Using Mathematics*, and the courses MU120 *Open Mathematics* and MS221 *Exploring Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MST121 uses the software program Mathcad (MathSoft, Inc.) and other software to investigate mathematical and statistical concepts and as a tool in problem solving. This software is provided as part of the course.

Contents

There are five sections in this chapter. They are intended to be studied consecutively in five study sessions. Subsection 2.2 requires the use of an audio cassette player, and Section 5 requires the use of the computer and Computer Book A.

The pattern of study for each session might be as follows.

Study session 1: Section 1

Study session 2: Section 2

Study session 3: Section 3

Study session 4: Section 4

Study session 5: Section 5

Each of the sections should take two and a half to three hours, the longest being Section 4.

Before studying this chapter, you should be familiar with the following topics, which are covered in the Revision Pack and Chapter A0:

◇ basic properties of powers a^x, where $a > 0$ and x is a real number;

◇ the notion of a function, and, in particular, trigonometric and exponential functions.

> The optional Video Band A(v) *Algebra workout – Powers and logarithms* could be viewed at any stage during your study of this chapter.
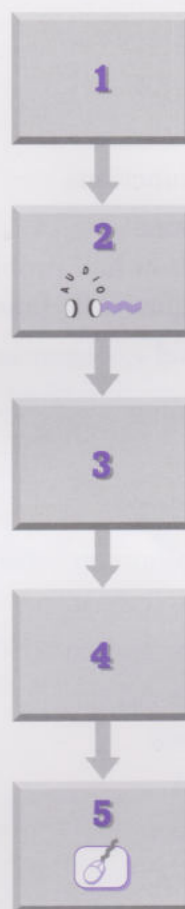
Introduction

One of the major concerns of mathematics (and science) is the way in which variables are related. For example, the formula

$$C = 2\pi r$$

expresses the circumference C of a circle in terms of its radius r. In this formula, r is an independent variable whereas C is a dependent variable. In more complicated formulas, there may be several independent variables. For example, in the formula $A = \frac{1}{2}ab\sin\theta$ for the area of a triangle, a, b and θ are independent variables, and A is the dependent variable; see Figure 0.1. In order to study the relationship between dependent and independent variables systematically, the concept of a *function* is introduced.

Figure 0.1 Area of a triangle

Section 1 introduces functions as 'processors' and describes their notation and their representation by graphs. It also introduces the concept of an *interval* of the real line; roughly speaking, this is an unbroken set of real numbers.

In Section 2, you will see how problems about areas lead to equations whose solutions can be estimated by using the graph of an appropriate function. We discuss techniques of translation and scaling for sketching graphs of functions, particularly quadratic functions.

Section 3 is concerned with trigonometric functions and exponential functions. You will see how fundamental properties of these functions are related to the shapes of their graphs.

If the effect of a given function can be reversed, then we obtain its *inverse function*. Section 4 defines this concept, explains how to identify inverse functions, and describes various standard inverse functions.

Section 5 uses the computer to plot accurate graphs of functions and to obtain numerical solutions of equations for which there is no convenient formula. It also describes how the computer can solve some equations by performing algebraic manipulations.

1 What is a function?

1.1 Function notation

In this chapter, we concentrate on functions which represent the relationship between a dependent variable and a single independent variable. In this case, it is common to take x as the independent variable (though other variables may be used, such as t when time is involved) and y as the dependent variable. So y and x are typically related by formulas such as

$$y = 2x - 1, \quad y = x^2, \quad y = \sqrt{x}, \quad y = \tan x, \quad y = 2^x.$$

In each case, for each value of x, the formula gives exactly one value of y – a unique value. If y and x are related by such a formula, then we say informally that 'y is a function of x'. The concept of a function is, however, more general than this, covering cases in which there is no simple formula relating y to x.

In general, a function can be thought of as a *processor* which converts inputs (values of x) into outputs (values of y). Figure 1.1 illustrates this concept in relation to the formula $y = x^2$.
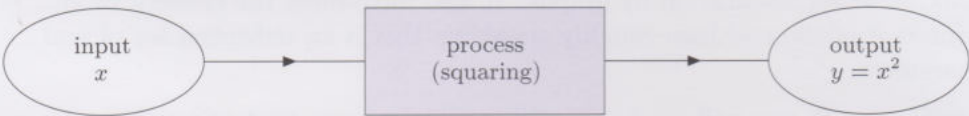
Figure 1.1 A processor for squaring

In general, a **function** is specified by giving:

(a) a set of allowed input values, called the **domain** of the function;

(b) a process, called the **rule** of the function, for converting each input value into a *unique* output value.

For a function called f, we use the notation $f(x) = \ldots$ to specify the rule of f, so $y = f(x)$. For example, for the function in Figure 1.1, the rule is

$$f(x) = x^2. \tag{1.1}$$

It is important to note that a function f is independent of the variable names, which could be changed *without* changing the function itself. For example, the formula $s = t^2$ also leads to the function f, with $s = f(t) = t^2$.

Various notations can be used to specify both the rule of a function *and* its domain, the commonest being of the form

$$f(x) = x^2 + 1 \quad (0 \le x \le 6). \tag{1.2}$$

Equation (1.2) specifies a function f with rule $f(x) = x^2 + 1$, whose domain is the set of real numbers x satisfying $0 \le x \le 6$.

Activity 1.1 Specifying functions

(a) Describe the domain of the function f given by the specification

$$f(t) = -t \quad (-1 < t < 2).$$

(b) For each of the following functions f, write down a specification of the function and find the value $f(1)$, the output corresponding to the input 1.

   (i) The function f which converts any non-negative real number x into x^2.

   (ii) The function f which converts any real number x which satisfies $-1 \le x \le 1$ into $2x + 1$.

Solutions are given on page 50.

For any x in the domain of f, we call $f(x)$ the **image** of x under f, the **value** of f at x, or simply an **image value**. For example, for the function in Figure 1.1, the image of -3 under f is $f(-3) = (-3)^2 = 9$.

The set of allowed input values of a function, the domain, depends on:

◇ the nature of the rule of the function – for example, the rule in equation (1.1) can be applied to any real number x, whereas the rule

$$f(x) = \sqrt{x}$$

can be applied only to *non-negative* real numbers;

◇ the context in which the function is applied – for example, if we are using the function with rule $f(x) = x^2$ in a modelling context, to determine the area of a square of side x, say, then it is inappropriate to include non-positive values of x in the domain.

Thus the domain of a function may be restricted by the nature of the rule of the function or by the context in which the function is applied.

In describing domains of functions, it is often convenient to use *interval notation*. Roughly speaking, an interval is an unbroken set of real numbers – that is, a 'subset' of the real line which can be 'drawn without lifting your pen from the paper'.

We usually use 'image', but as you will see, there are contexts where the alternatives are more natural.

A subset of the real line that is *not* an interval is the 'broken' subset comprising the digits $0, 1, \ldots, 9$.

Intervals

Let a and b be real numbers, with $a < b$.

The **closed interval** $[a, b]$ is the set of real numbers x such that $a \le x \le b$.

The **open interval** (a, b) is the set of real numbers x such that $a < x < b$.

Some texts use the notation $]a, b[$ for open intervals.

The closed interval $[a, b]$ includes its *endpoints* a and b, whereas the open interval (a, b) excludes them. For example, in part (ii) of Activity 1.1(b), the domain of the function is the closed interval $[-1, 1]$.

Interval notation may be used when specifying functions. For example, rather than writing

$$f(x) = 2x + 1 \quad (-1 \le x \le 1)$$

to specify the function in part (ii) of Activity 1.1(b), we could write

$$f(x) = 2x + 1 \quad (x \text{ in } [-1, 1]).$$

We can also use this notation to represent intervals which include one endpoint but exclude the other. Also, by using the symbols ∞ and $-\infty$, we can represent intervals which extend indefinitely far to the right or left; see Table 1.1 (overleaf).

Read ∞ and $-\infty$ as 'infinity' and 'minus infinity'.

Table 1.1   Interval notation

	Inequality	Interval
Closed intervals	$a \leq x \leq b$	$[a, b]$
	$a \leq x$	$[a, \infty)$
	$x \leq a$	$(-\infty, a]$
Open intervals	$a < x < b$	(a, b)
	$a < x$	(a, ∞)
	$x < a$	$(-\infty, a)$
Half-open (or half-closed)	$a \leq x < b$	$[a, b)$
intervals	$a < x \leq b$	$(a, b]$

Remember that the symbols ∞ and $-\infty$ do *not* represent real numbers, so they are not considered to be endpoints of intervals.

The real line $\mathbb{R}$ is also an interval, sometimes written as $(-\infty, \infty)$.

Activity 1.2   Using interval notation

For each of the following inequalities, write down the corresponding interval and describe it as closed, open or half-open (half-closed).

(a) $0 < x < 1$ (b) $-3 \leq x \leq 2$ (c) $-2 < x \leq 2$ (d) $x \geq 0$

Solutions are given on page 50.

When the domain of a function is restricted only by the nature of the rule of the function, we usually do not state the domain explicitly, but rely on the following convention.

> **Domain convention**
>
> When a function is specified by *just a rule*, it is understood that the domain of the function is the largest possible set of real numbers for which the rule is applicable.

Consider, for example, the two functions g and h specified as follows:

$$g(x) = \sqrt{x} \quad \text{and} \quad h(x) = \frac{1}{x}.$$

By the domain convention, the function g has domain $[0, \infty)$, since $\sqrt{x}$ is defined only for $x \geq 0$. Similarly, the function h has domain $\mathbb{R}$ excluding 0, which consists of the two open intervals $(-\infty, 0)$ and $(0, \infty)$.

Activity 1.3   Using the domain convention

Specify the domain of each of the following functions, as given by the domain convention. In each case, explain your answer.

(a) $f(x) = \sqrt{x - 1}$

(b) $f(x) = \dfrac{1}{x - 2} + \dfrac{1}{x + 3}$

Solutions are given on page 50.

Here are some general remarks about function notation.

1. As you have seen, it is common practice to use the name f for the function currently being discussed. Other names like g and h are used in order to distinguish between functions.

2. Alternative names for 'function' are *mapping* and, in geometric contexts, *transformation*.

3. There is an alternative notation for the rule of a function. Rather than writing $f(x) = x^2$, for example, we can write

    $$f : x \longmapsto x^2.$$

    This notation aims to suggest the sense of x being converted ('mapped') to x^2 by the function.

    It is also common practice to write, for example, 'the function $f(x) = x^2$' rather than 'the function f with rule $f(x) = x^2$'.

    > Read $f : x \longmapsto x^2$ as 'the function f maps x to x^2'.

4. For all the functions in this chapter, both the inputs and the outputs are real numbers. Such functions are called **real functions**. More general functions may have inputs or outputs of a very different nature, such as points in the plane, words or even people. When dealing with such general functions, it is common to add to the function specification information about a set in which the output values lie. We call such a set the **codomain** of the function. For example, to specify the function $f(x) = \sqrt{x}$ in this detailed way, we could write

    $$f : [0, \infty) \longrightarrow \mathbb{R}$$
    $$x \longmapsto \sqrt{x}.$$

    Notice that two different types of arrow are used here.

    > Here $[0, \infty)$ is the domain and $\mathbb{R}$ is the codomain. For a real function, the codomain can always be chosen to be $\mathbb{R}$.

1.2   Graphs of functions

For any real function f, the points (x, y) in the Cartesian plane which satisfy $y = f(x)$ form the **graph** of the function, often referred to as 'the graph of $y = f(x)$'. Such a graph gives a means of visualising the function geometrically, often illustrating key features of the function. For example, in Chapter A2 you met straight-line graphs with equations of the form $y = mx + c$. These are the graphs of **linear functions** of the form

$$f(x) = mx + c.$$

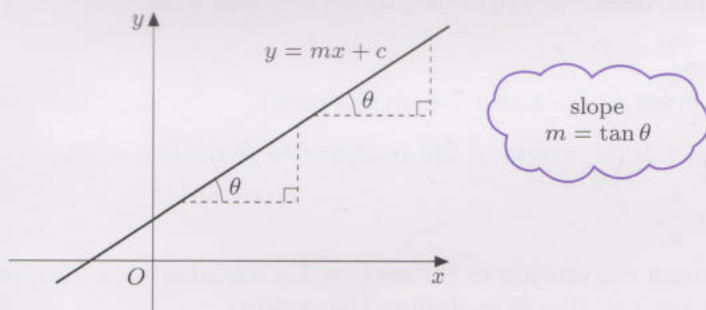> Here the domain convention is being used.

Figure 1.2   Graph of $y = mx + c$

The fact that the graphs of these functions are (straight) lines means that they have constant slope, and this in turn means that if x increases by a fixed amount, then so does y; see Figure 1.2.

Graphs of linear functions can be drawn accurately by plotting just two points and drawing a line passing through them. For most other functions, however, we can sketch graphs only approximately.

A basic method is to construct a table of values of $f(x)$ for several values of x, plot the corresponding points $(x, f(x))$, and attempt to sketch a smooth curve through these points. The more points we can plot, the better the sketch will be, and the more confident we shall be that the function does not behave wildly between consecutive plotted points. One possibility is to use a graphics calculator or computer to do this, but first it is important to develop some familiarity with the graphs of a number of common types of functions.

For most of the functions considered in this chapter, the graph will be smooth.

We begin with the 'squaring function'

$$f(x) = x^2,$$

and sketch the corresponding graph of $y = x^2$ by constructing a table of values such as the following.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
x^2	4	2.25	1	0.25	0	0.25	1	2.25	4

Where convenient, we use the same scale on both axes of a graph, as here.
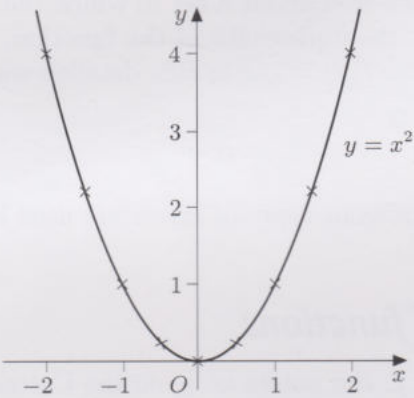

Figure 1.3   Graph of $y = x^2$

In Figure 1.3 we have plotted 9 points and joined them by a smooth U-shaped curve. This characteristic smooth U-shape can be confirmed by plotting more points if necessary. The curve is symmetric in the y-axis; that is, the part to the left of the y-axis is the mirror image, or reflection, in the y-axis of the part to the right. This is because if the point (u, v) lies on the graph, then $v = f(u) = u^2$, so we can also write

For example, the points $(2, 4)$ and $(-2, 4)$ both lie on the graph.

$$v = (-u)^2 = f(-u).$$

Hence the point $(-u, v)$ also lies on the graph.

Next, we sketch the graph of the **reciprocal function**

Recall that, for $x \neq 0$, the reciprocal of x is $1/x$.

$$f(x) = \frac{1}{x}.$$

By the domain convention in Subsection 1.1, the domain of the reciprocal function is the real line $\mathbb{R}$ excluding the point 0.

As before, we construct a table of values for $f(x) = 1/x$, and try to sketch a smooth curve (or curves) through the corresponding points $(x, f(x))$.

These values of $1/x$ are correct to two decimal places.

x	-2	-1.5	-1	-0.5	0.5	1	1.5	2
$1/x$	-0.5	-0.67	-1	-2	2	1	0.67	0.5

Observe from the table that points on the graph of f lie in either the first or the third quadrant. Since 0 is not in the domain of f, no point in the first quadrant is joined by the graph to a point in the third quadrant. The graph is shown in Figure 1.4.
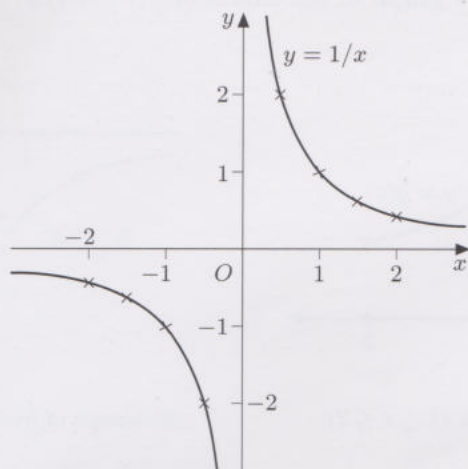

Figure 1.4   Graph of $y = 1/x$

In this case, using the same scale on both axes suggests that the graph is symmetric in the line $y = x$, as discussed below.

Figure 1.4 shows that the graph consists of *two* smooth curves, each of which approaches – that is, becomes arbitrarily close to – an axis at each end. This behaviour of the graph occurs because, for $x > 0$,

    if x is large, then $f(x) = \dfrac{1}{x}$ is small,

and

    if x is small, then $f(x) = \dfrac{1}{x}$ is large,

For example, if $x = 1000$, then $1/x = 10^{-3}$.

with similar properties for $x < 0$.

Any line which a curve approaches 'far from the origin' is called an **asymptote**. Thus both the axes are asymptotes of the graph of the reciprocal function.

The graph of $y = 1/x$ also has some symmetry properties, which can be seen in the points plotted in Figure 1.4. For example, the points $(2, 0.5)$, $(-2, -0.5)$, $(0.5, 2)$ and $(-0.5, -2)$ all lie on the graph. In general, if the point (u, v) lies on the graph, then so do the points $(-u, -v)$, (v, u) and $(-v, -u)$; see Figure 1.5.
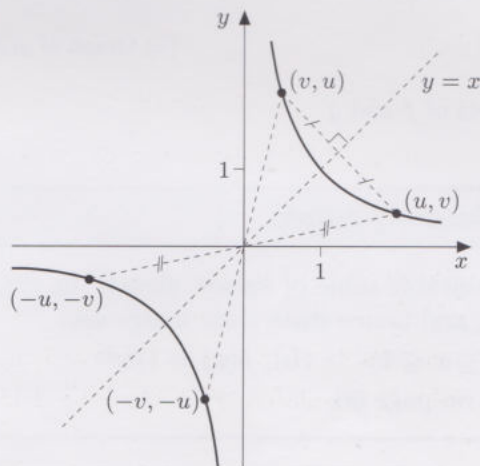

Figure 1.5   Symmetry properties of the graph of $y = 1/x$

As in Figure 1.5, it is often sufficient to show just two axis scale markers to indicate the scales being used for a graph, and to omit the label for the origin unless the origin has a special role (such as an intercept) in the graph under consideration.

The idea of restricting the domain of a function was introduced on page 7.

If the domain of a function f has been restricted due to the context in which the function is applied, then the graph of f is restricted in a corresponding way. For example, the graphs of the functions
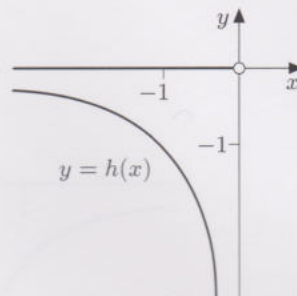
$$g(x) = 1/x \quad (1 \le x \le 2) \quad \text{and} \quad h(x) = 1/x \quad (-\infty < x < 0)$$

are both parts of the graph of the function $f(x) = 1/x$; see Figure 1.6.

Here, in each case, the domain on the x-axis has been emphasised. To indicate whether a point is included or excluded from a domain or graph, we can draw it as a solid dot (included) or an open circle (excluded).
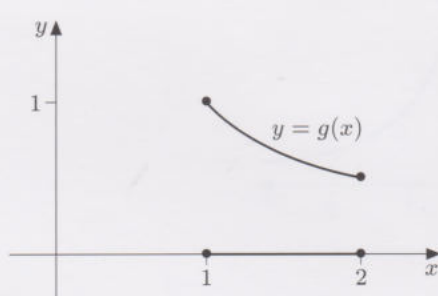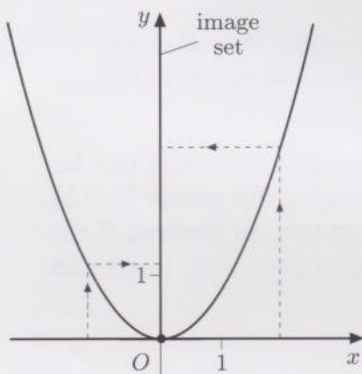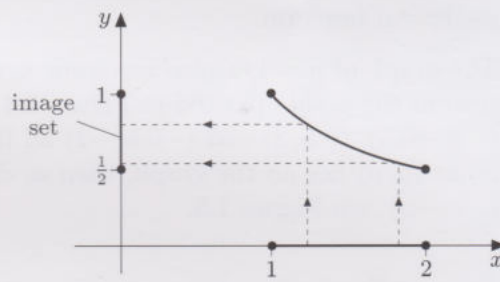

(a) Graph of $g(x) = 1/x$ $(1 \le x \le 2)$ (b) Graph of $h(x) = 1/x$ $(-\infty < x < 0)$

Figure 1.6   Graphs of g and h

One feature of a function f which can often be read off from the graph of f is the **image set**, or **image**, of f. The image set is the complete set of image values of f – that is, all possible values $f(x)$, where x lies in the domain of f. In terms of the graph of f, the image set of f is the set of points on the y-axis which are at the same height as a point of the graph. Figure 1.7 shows the image sets of two of the functions considered above. The image set of the function $f(x) = x^2$ is the interval $[0, \infty)$, and the image set of the function $g(x) = 1/x$ $(1 \le x \le 2)$ is the interval $[\frac{1}{2}, 1]$.

(a) Graph of $f(x) = x^2$ (b) Graph of $g(x) = 1/x$ $(1 \le x \le 2)$

Figure 1.7   Image sets of f and g

Activity 1.4   Sketching graphs

Calculate image values correct to two decimal places.

By constructing a suitable table of values, sketch the graphs of each of the following functions, and hence state their image sets.

(a) $f(x) = x^3$ $(-1 \le x \le 1)$ (b) $f(x) = 1/x^2$

Solutions are given on page 50.

So far, the graphs considered have consisted of a smooth curve (or curves). The next graph is smooth except at one point, where it turns a corner! This is the graph of the *modulus function*.

> The **modulus**, or **absolute value**, of a real number x is the magnitude of x, regardless of its sign; it is denoted by $|x|$.

For example:

the modulus of -3 is $|-3| = 3$;

the modulus of 0 is $|0| = 0$;

the modulus of π is $|\pi| = \pi$.

From this definition, it follows that

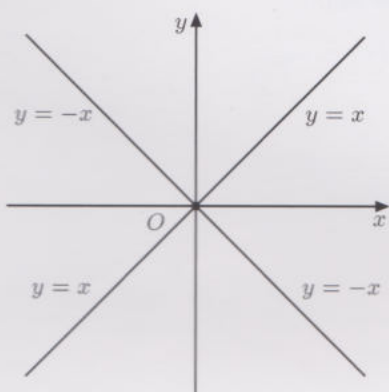$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

For example,
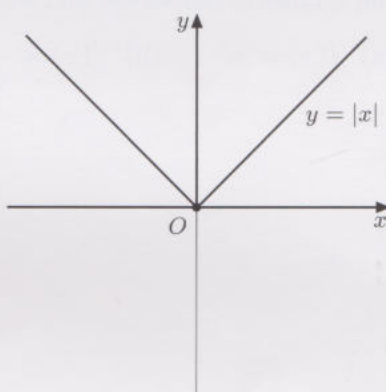
$$|-3| = -(-3) = 3.$$

This observation enables us to sketch the graph of the **modulus function**

$$f(x) = |x|$$

without constructing a table of values. We just use the part of the graph of $y = x$ for which $x \geq 0$, and the part of the graph of $y = -x$ for which $x < 0$, as shown in Figure 1.8.

(a) Graphs of $y = x$ and $y = -x$ (b) Graph of $y = |x|$

Figure 1.8   Constructing the graph of the modulus function

The resulting graph of $y = |x|$ has a corner at the origin, and its image set is $[0, \infty)$.

Activity 1.5   Graphs involving the modulus

Without constructing a table of values, sketch the graphs of each of the following functions.

(a) $f(x) = |x|^3 \ (-1 \leq x \leq 1)$ (b) $f(x) = \dfrac{1}{|x|}$

Solutions are given on page 50.

Summary of Section 1

This section has introduced:

◇   notations for specifying the rule and domain of a function;

◇   interval notation, including $[a, b]$ for a closed interval and (a, b) for an open interval;

◇   a domain convention for functions;

◇   the graph and image set of a function;

◇   the graphs of some common functions;

◇   the modulus, or absolute value, of a real number.

Exercises for Section 1

Exercise 1.1

For each of the following functions, write down the domain of the function using interval notation.

(a) $f(x) = x^4 \ (x < 1)$ (b) $f(x) = \dfrac{1}{\sqrt{x}}$ (c) $f(x) = \sqrt{9 + x}$

Exercise 1.2

Calculate image values correct to two decimal places.

By constructing tables of values, or otherwise, sketch the graphs of each of the following functions, and state their image sets.

(a) $f(x) = x^4$ (b) $f(x) = \dfrac{1}{\sqrt{x}}$ (c) $f(x) = \dfrac{1}{\sqrt{|x|}}$

2 Quadratic functions

To study Subsection 2.2, you will need an audio cassette player and Audio Tape 1.

In Section 1, we discussed the idea of a (real) function, and considered some examples of common functions. Now we see how functions and their graphs can be applied in a particular modelling situation.

2.1 The exhibition hall problem

A hall is to be used for an exhibition in such a way that half the floor space is devoted to exhibits, the rest being clear. After measuring the hall and making simplifying assumptions (such as taking convenient approximate dimensions), the following problem is stated.

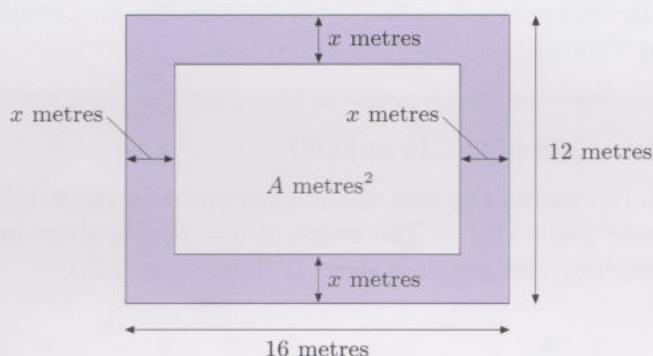> **The Exhibition Hall Problem**
>
> A rectangular hall of dimensions 16 metres by 12 metres is to have a uniform border around the walls for exhibits (as shown in Figure 2.1). The border is to take half the area of the hall. Determine the width of the border.

In Figure 2.1, the variable x has been used to denote the width of the border, and the variable A denotes the area of the clear space.

From now on we assume that all lengths are in metres.

Figure 2.1 Introducing variables for the exhibition hall problem

To solve the exhibition hall problem, we need to find the value of x for which A is half the total area of the hall. You are now asked to find a formula for A in terms of x.

Activity 2.1 Relating variables

(a) Write down the largest *closed* interval consisting of values of x which make sense for the exhibition hall problem.

(b) For values of x in this interval, write down expressions for the length and width of the clear space.

(c) Deduce an expression for the area of the clear space A, in terms of x, multiplying out any brackets.

Solutions are given on page 51.

In Activity 2.1, you should have found that x lies in the interval $[0, 6]$, and that, for such x,

$$A = 4x^2 - 56x + 192.$$

The exhibition hall problem is to find a value of x such that A is half the area of the hall; that is, $A = \frac{1}{2} \times 16 \times 12 = 96$. Therefore we have to solve the equation $4x^2 - 56x + 192 = 96$; that is,

$$4x^2 - 56x + 96 = 0. \tag{2.1}$$

Then we have to choose a solution in the interval $[0, 6]$, if possible.

Activity 2.2 Solving the equation

(a) Solve equation (2.1).

(b) Hence solve the exhibition hall problem.

Solutions are given on page 51.

Comment

You can check whether your answer to part (b) is reasonable by studying Figure 2.1, which has been drawn (to scale) with $A = 96$.

Interpret results

We have now solved the exhibition hall problem, and without any reference to functions or graphs! What makes this problem straightforward is the existence of a convenient method for solving equation (2.1). For many problems, however, we arrive at equations for which no such method is available. In such cases, it *is* helpful to express the problem in terms of functions and their graphs.

For example, a function which arises in the exhibition hall problem is

$$f(x) = 4x^2 - 56x + 192 \quad (x \text{ in } [0, 6]).$$

This function represents the area of the clear space in the exhibition hall, when the border has width x. The graph of $y = f(x)$ is given in Figure 2.2. (A way of sketching this graph is given in Subsection 2.2.)

For this graph, it is *not* convenient to use the same scale on both axes, because the domain of f is $[0, 6]$ and the image set of f is $[0, 192]$.
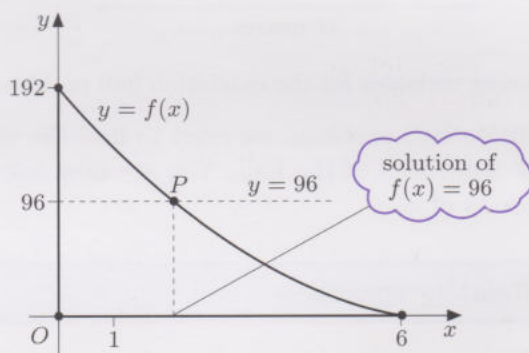

Figure 2.2 Graph of $f(x) = 4x^2 - 56x + 192$ (x in $[0, 6]$)

The graph of f has the following features:

◇ $f(0) = 192$ (if $x = 0$, then the hall is completely clear);

◇ $f(6) = 0$ (if $x = 6$, then there is no clear space);

◇   as x increases from 0 to 6, the clear area $f(x)$ decreases from 192 to 0.

The key point to notice from Figure 2.2 is that it provides some indication of the answer to the exhibition hall problem. The answer to this problem is a solution of the equation $f(x) = 96$; that is, it is the value of x at the point P where the horizontal line $y = 96$ meets the graph of $y = f(x)$. From Figure 2.2, we can see that this value of x is approximately 2, which is the value found in the solution to Activity 2.2(b).

The graph of the same function f can also be used to obtain an approximate value of the solution to a *modified* exhibition hall problem. Suppose that the area of the clear space is required to be $\frac{3}{4}$ of the total area (192 square metres); that is, $A = \frac{3}{4} \times 192 = 144$. Then the border width x must satisfy

$$f(x) = 144. \tag{2.2}$$

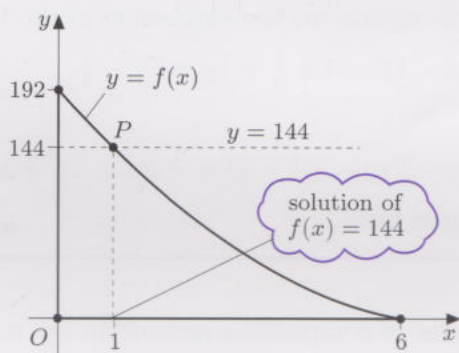It can be seen from Figure 2.3 that in this case x is approximately 1.

Figure 2.3   Modified exhibition hall problem

Activity 2.3   The modified exhibition hall problem

(a) Solve equation (2.2).

(b) Hence solve the modified exhibition hall problem.

Solutions are given on page 51.

So, expressing a problem in terms of a function and its graph can provide an approximation to the solution to the problem. A graph can also convey more general information. For example, if the area of clear space in the exhibition hall is required to be *any* particular value, A say, in the image set $[0, 192]$, then there is a unique corresponding value of x for which this area occurs, namely, the solution in $[0, 6]$ of the equation $f(x) = A$.

In Section 5, we use graphs in this way to help solve equations for which there is no convenient formula.

2.2 Graphs of quadratic functions

A **quadratic function** is one whose rule is of the form

$$f(x) = ax^2 + bx + c \quad \text{(where } a \neq 0\text{)}.$$

In the audio tape which follows, techniques of *translation* and *scaling* are introduced. These techniques can be used to sketch the graph of any quadratic function, without constructing a table of values, by relating the required graph to the familiar graph of $y = x^2$ in a simple geometric way.   See Subsection 1.2.

To prepare for these new graph-sketching techniques, you are reminded of a technique, 'completing the square', which is based on the identity

$$x^2 + 2px = (x + p)^2 - p^2. \tag{2.3}$$

Using equation (2.3), any quadratic expression $ax^2 + bx + c$ can be rearranged in *completed-square form* as $a(x + p)^2 + q$.

Example 2.1   Completing the square

Rearrange $-2x^2 + 4x - 1$ in completed-square form.

Solution

In order to use equation (2.3), we first take out the factor -2:

$$-2x^2 + 4x - 1 = -2(x^2 - 2x + \tfrac{1}{2}).$$

Next, we complete the square for the expression in the bracket:

Since the coefficient of x is -2, we take $p = -1$ in equation (2.3).

$$x^2 - 2x + \tfrac{1}{2} = (x - 1)^2 - 1 + \tfrac{1}{2} = (x - 1)^2 - \tfrac{1}{2}.$$

Thus we obtain

$$-2x^2 + 4x - 1 = -2((x - 1)^2 - \tfrac{1}{2}) = -2(x - 1)^2 + 1,$$

which is in the required form.

The following examples of completed-square form are used on the tape.

Activity 2.4   Completing the square

Rearrange each of the following expressions in completed-square form.

(a) $x^2 + 4x + 3$ (b) $x^2 + x + \frac{3}{4}$ (c) $4x^2 - 56x + 192$ (d) $\frac{1}{2}x^2 + x$
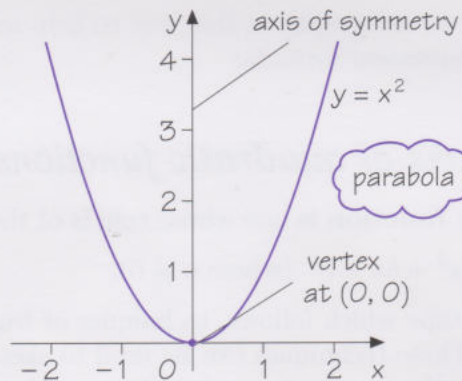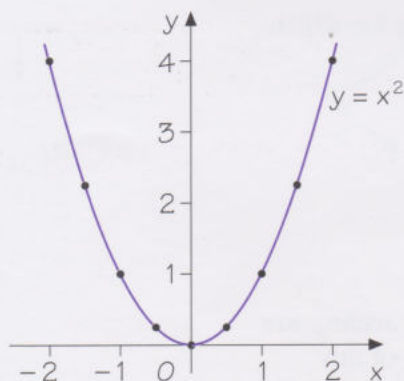
Solutions are given on page 51.

Now listen to Audio Tape 1, Band 2, 'Graphs of quadratic functions'.
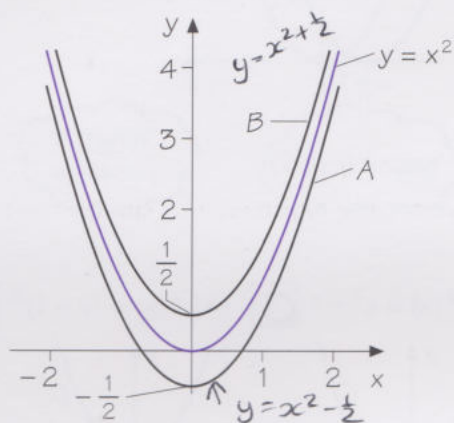
Frame 1

The graph of the function $f(x) = x^2$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
x^2	4	2.25	1	0.25	0	0.25	1	2.25	4

Frame 2

The graph of the function $f(x) = x^2 + \frac{1}{2}$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
x^2	4	2.25	1	0.25	0	0.25	1	2.25	4
$x^2 + \frac{1}{2}$	4.5	2.75	1.5	0.75	0.5	0.75	1.5	2.75	4.5

Which curve is
$y = x^2 + \frac{1}{2}$, B
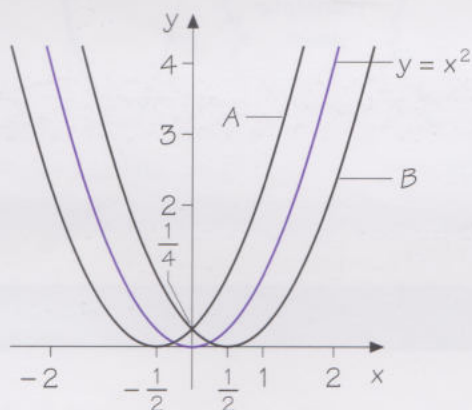A or B ?

Vertex: $\left(0, \frac{1}{2}\right)$

Axis of symmetry: $x = 0$

Frame 3

The graph of the function $f(x) = \left(x + \frac{1}{2}\right)^2$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
x^2	4	2.25	1	0.25	0	0.25	1	2.25	4
$x + \frac{1}{2}$	-1.5	-1.0	-0.5	0	0.5	1.0	1.5	2.0	2.5
$\left(x + \frac{1}{2}\right)^2$	2.25	1.0	0.25	0	0.25	1.0	2.25	4.0	6.25

Which curve is
$y = \left(x + \frac{1}{2}\right)^2$, A
A or B ?

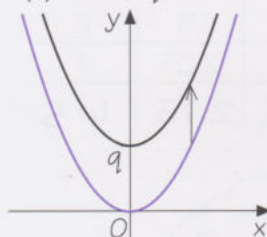Vertex: $\left(-\frac{1}{2}, 0\right)$

Axis of symmetry: $x = -\frac{1}{2}$
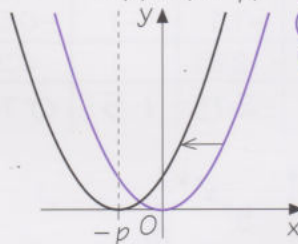
Frame 4

Translations

$f(x) = x^2 + q$

Translate up by q

Vertex: $(0, q)$
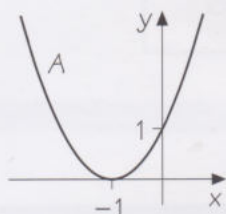
Down if q negative

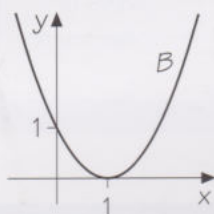$f(x) = (x + p)^2$

Translate to left by p

Vertex: $(-p, 0)$
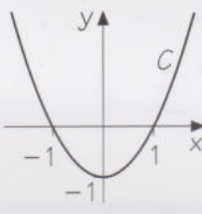
To right if p negative
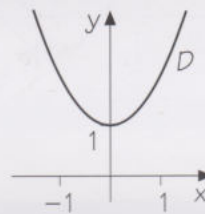
Match each function to its graph.

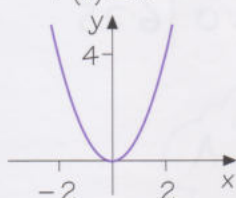(1) $f(x) = x^2 + 1$ $\boxed{D}$　　(2) $f(x) = (x + 1)^2$ $\boxed{A}$　　(3) $f(x) = x^2 - 1$ $\boxed{C}$　　(4) $f(x) = (x - 1)^2$ $\boxed{B}$
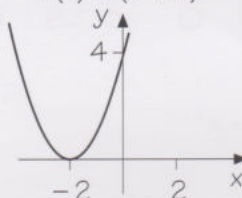
Frame 5

The function $f(x) = (x + 2)^2 - 1$

$f(x) = x^2$
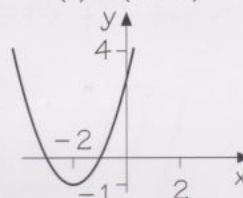
Vertex: $(0, 0)$

Translate to left by 2

$f(x) = (x + 2)^2$

Vertex: $(-2, 0)$

Translate down by 1

$f(x) = (x + 2)^2 - 1$
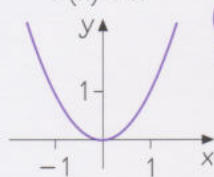
Vertex: $(-2, -1)$

Frame 6

The function $f(x) = \left(x + \frac{1}{2}\right)^2 + \frac{1}{2}$

$f(x) = x^2$

Vertex: $(0, 0)$

Translate $\boxed{\text{left}}$ by $\boxed{\frac{1}{2}}$

$f(x) = \left(x + \frac{1}{2}\right)^2$
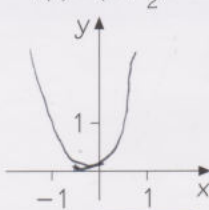
Vertex: $\boxed{\left(-\frac{1}{2}, 0\right)}$

Translate $\boxed{\text{up}}$ by $\frac{1}{2}$

$f(x) = \left(x + \frac{1}{2}\right)^2 + \frac{1}{2}$
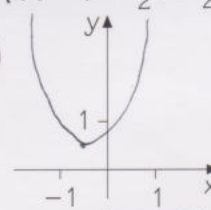
Vertex: $\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$

Frame 7

The function $f(x) = ax^2$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
x^2	4	2.25	1	0.25	0	0.25	1	2.25	4
$2x^2$	8	4.50	2	0.50	0	0.50	2	4.50	8
$\frac{1}{2}x^2$	2	1.125	0.50	0.125	0	0.125	0.5	1.125	2
$-x^2$	-4	-2.25	-1	-0.25	0	-0.25	-1	-2.25	-4
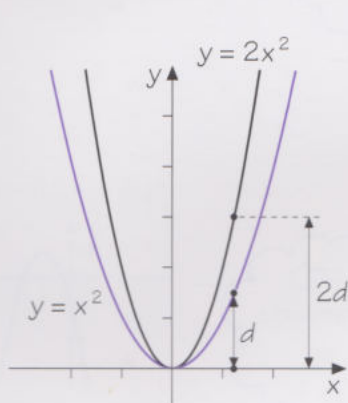
y-scaling
with factor 2
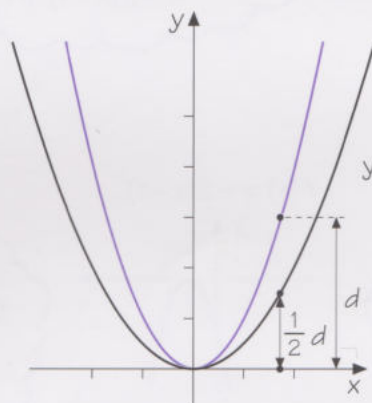
y-scaling
with factor $\frac{1}{2}$
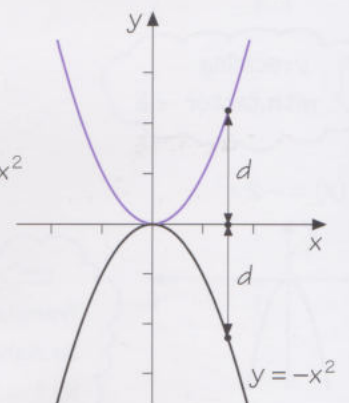
y-scaling
with factor -1

Frame 8

Graph of the parabola $y = ax^2$

$a > 1$ $a = 1$ (basic parabola)

$0 < a < 1$
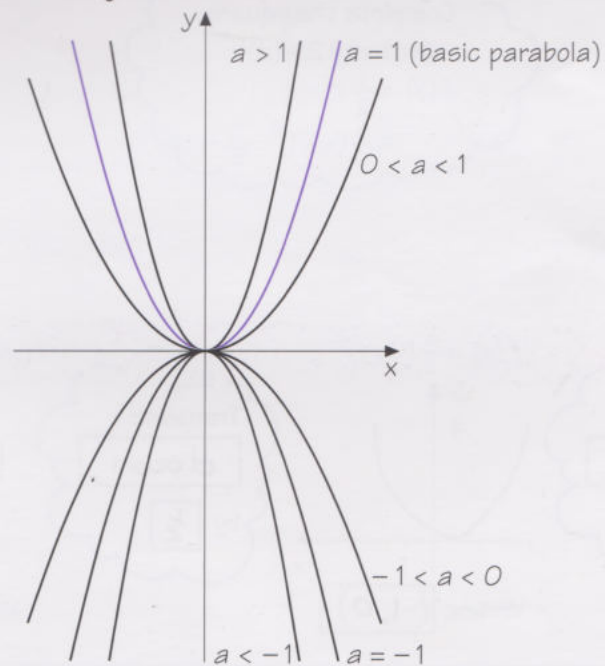
y-scaling
by factor a

$-1 < a < 0$

$a < -1$ $a = -1$

Frame 9

The function $f(x) = -2x^2 + 4x - 1$

$f(x) = x^2$
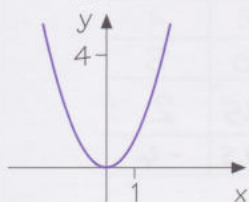
> Complete the square
> (Example 2.1):
> $f(x) = -2x^2 + 4x - 1$
> $\qquad = -2(x-1)^2 + 1$

> y-scaling
> with factor -2

$f(x) = -2x^2$
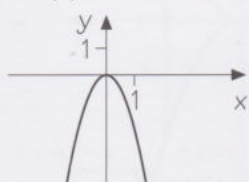
Vertex: $(0, 0)$

> Translate
> to right
> by 1

$f(x) = -2(x-1)^2$
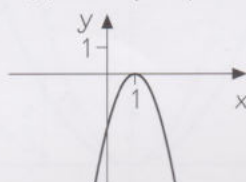
Vertex: $(1, 0)$

> Translate
> up
> by 1

$f(x) = -2(x-1)^2 + 1$
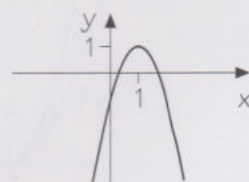
Vertex: $(1, 1)$

Frame 10

The function $f(x) = \frac{1}{2}x^2 + x$

$f(x) = x^2$
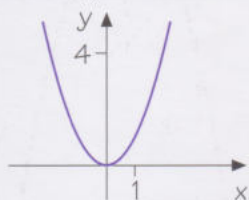
> Complete the square
> (Activity 2.4(d)):
> $f(x) = \frac{1}{2}x^2 + x$
> $\qquad = \frac{1}{2}(x+1)^2 - \frac{1}{2}$

> y-scaling
> with factor $\boxed{\frac{1}{2}}$

$f(x) = \frac{1}{2}x^2$
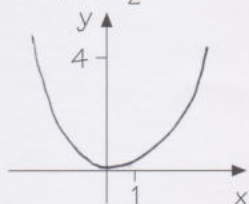
Vertex: $\boxed{(0, 0)}$

> Translate
> $\boxed{\text{left}}$
> by $\boxed{1}$

$f(x) = \frac{1}{2}(x+1)^2$
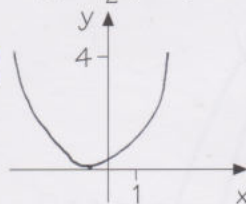
Vertex: $\boxed{(-1, 0)}$

> Translate
> $\boxed{\text{down}}$
> by $\boxed{\frac{1}{2}}$

$f(x) = \frac{1}{2}(x+1)^2 - \frac{1}{2}$
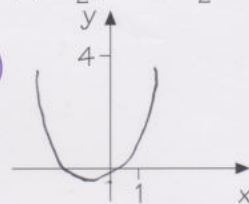
Vertex: $\boxed{\left(-1, -\frac{1}{2}\right)}$

Frame 11

Summary : $f(x) = a(x + p)^2 + q$

	a positive	a negative
$a = 1$		
a small $(-1 < a < 1)$		
a large $(a < -1, a > 1)$		

Vertex: $(-p, q)$

Axis of symmetry: $x = -p$

Here are some further remarks about sketching quadratic graphs.

1. Another useful technique to use when sketching the graph of a function is to check where the graph meets the axes – that is, to find the x- and y-intercepts. For a quadratic function $f(x) = ax^2 + bx + c$,

   By the domain convention, f has domain $\mathbb{R}$.

   ◇   the y-intercept is found by putting $x = 0$ to give

   $$f(0) = c,$$

   ◇   the x-intercepts (if any) are found by solving the equation

   $$f(x) = ax^2 + bx + c = 0.$$

   A quadratic equation may have 0, 1 or 2 solutions.

   For example, in Frame 9, we sketched the graph of the function $f(x) = -2x^2 + 4x - 1$. In this case:

   ◇ $f(0) = -1$, so the y-intercept is -1;

   ◇   the equation $-2x^2 + 4x - 1 = 0$ has solutions $x = 1 \pm \frac{1}{2}\sqrt{2}$, so the x-intercepts are $x \simeq 0.29$ and $x \simeq 1.71$.

   These x- and y-intercepts are shown in Figure 2.4.

2. In both Frames 9 and 10, we first performed a y-scaling, followed by a horizontal and then a vertical translation. You may prefer, however, to reverse the order of the y-scaling and the horizontal translation. This approach is illustrated in Figure 2.5 for the function

   $$f(x) = -2x^2 + 4x - 1 = -2(x - 1)^2 + 1. \tag{2.4}$$

Figure 2.4   Graph of $y = -2x^2 + 4x - 1$
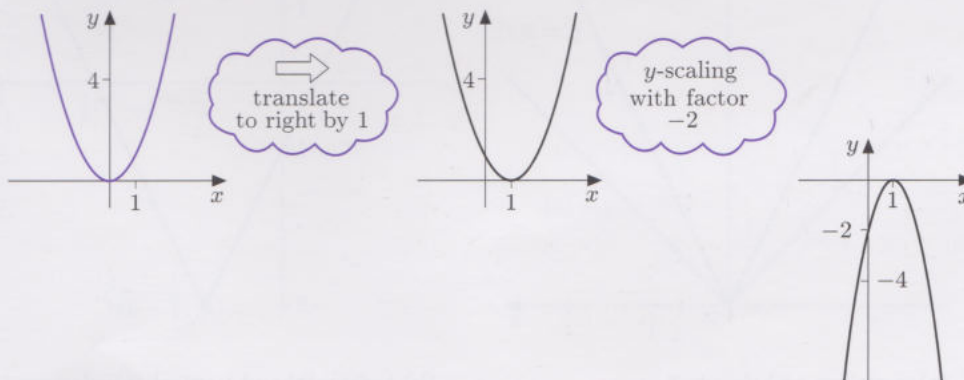
Figure 2.5   Horizontal translation before y-scaling

This vertical translation must be performed *last*, since $+1$ is the last operation performed when evaluating $f(x)$ using equation (2.4).

The vertical translation by 1 unit upwards then completes the process, as in Frame 9.

3. Even when the graph of the function f that you wish to sketch requires different scales on the axes in order to represent it, the translation and scaling techniques can still be used. Such situations may occur when large coefficients exist in the expression for $f(x)$, as in the case of the function

$$f(x) = 4x^2 - 56x + 192.$$

Some trial and error may be needed when finding suitable axis scales.

In such cases, it is necessary to choose scales so that any intercepts can be represented. Sketching the function f above is the subject of the next activity.

Use the techniques introduced in the audio tape to sketch the graph of the quadratic function

$$f(x) = 4x^2 - 56x + 192,$$

You rearranged this quadratic expression into completed-square form in Activity 2.4(c).

including the x- and y-intercepts.

A solution is given on page 51.

Comment

The graph of the function introduced in connection with the exhibition hall problem is *part* of the graph sketched in this activity; see Figure 2.2.

The techniques of translation and scaling can be used to help sketch the graphs of functions other than quadratic ones. For example, the graph of the function

$$f(x) = 2|x - 1| - 3$$

can be obtained from the graph of the modulus function $f(x) = |x|$ by performing:

◇   a y-scaling with factor 2;

◇   a horizontal translation by 1 unit to the right;

◇   a vertical translation by 3 units downwards.

The stages in this process are shown in Figure 2.6. It is convenient to perform the two translations together, as in Figure 2.6(b).

Here the graph of $y = |x|$ is shown in colour.
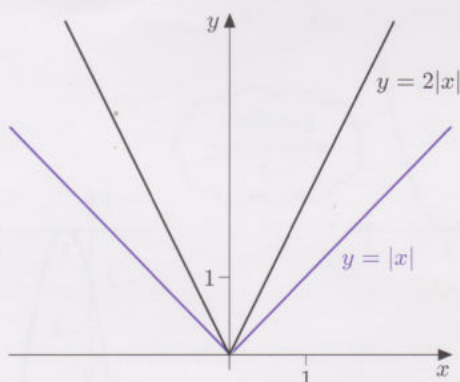
(a) y-scaling with factor 2
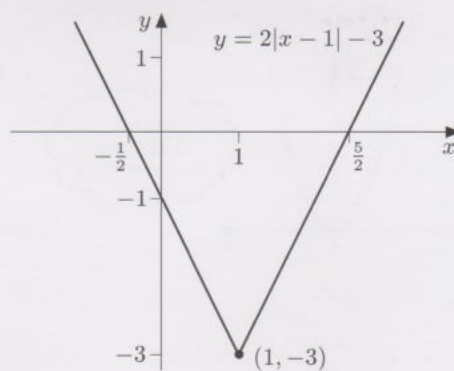
(b) horizontal and vertical translations

Figure 2.6   Constructing the graph of $y = 2|x - 1| - 3$

The graph of the function $f(x) = 2|x - 1| - 3$ in Figure 2.6(b) shows the y-intercept as

$$f(0) = 2|0 - 1| - 3 = 2 - 3 = -1,$$

and the x-intercepts as $x = -\frac{1}{2}$ and $x = \frac{5}{2}$. The x-intercepts are found by solving the equation

$$f(x) = 2|x - 1| - 3 = 0;$$

that is,

$$|x - 1| = \tfrac{3}{2}.$$

The equation $|x - 1| = \frac{3}{2}$ is equivalent to the two equations

$$x - 1 = \tfrac{3}{2} \quad \text{and} \quad x - 1 = -\tfrac{3}{2}.$$

Thus the solutions are $x = \frac{5}{2}$ and $x = -\frac{1}{2}$, confirming the x-intercepts given in the figure.

Another way to write these two equations is

$$x - 1 = \pm\tfrac{3}{2}.$$

In addition to horizontal and vertical translations and y-scalings of graphs, there are also x-scalings which correspond to squashing or stretching a graph in the x-direction by a given factor. For example, consider the effect of squashing the graph of the function $f(x) = x^2$ in the x-direction by the factor $\frac{1}{2}$. Let (u, v) be a point on the graph of f; see Figure 2.7(a). On what curve does the point $(\frac{1}{2}u, v)$ lie? Since

$$v = f(u) = u^2,$$

we have

$$v = u^2 = 4(\tfrac{1}{2}u)^2.$$

Thus the point $(\frac{1}{2}u, v)$ lies on the graph of $y = 4x^2$.

So if g is the function $g(x) = 4x^2$, then its graph can be obtained by squashing the graph of f in the x-direction by the factor $\frac{1}{2}$; see Figure 2.7(b).

The function g can be expressed in terms of f as

$$g(x) = 4x^2 = (2x)^2 = f(2x),$$

and the graph of g can be obtained from that of f by applying an x-scaling with factor $\frac{1}{2}$.

Alternatively, the graph of g can be obtained from that of f by means of a y-scaling with factor 4.

(a) Graph of $f(x) = x^2$ (b) Graphs of $f(x) = x^2$ and $g(x) = 4x^2$
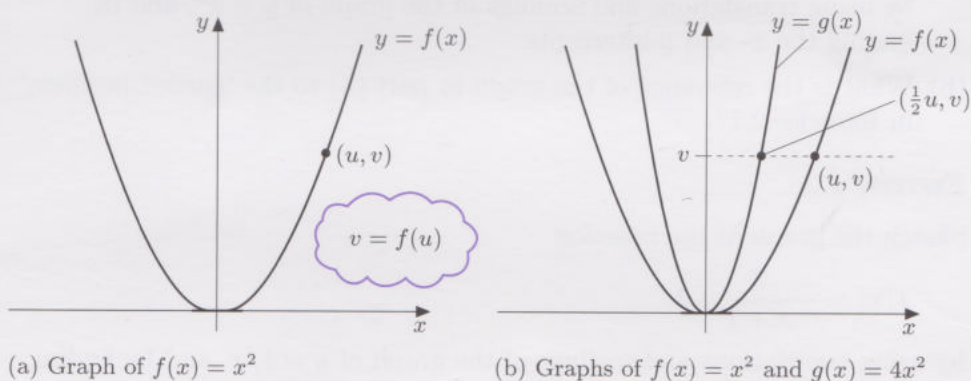
Figure 2.7   An x-scaling

More generally, the graph of $y = f(bx)$, where $b \neq 0$, is obtained from the graph of $y = f(x)$ by an x-**scaling** with factor $1/b$. You will meet some examples of the x-scaling of graphs in Section 3.

An x-scaling with factor -1 corresponds to reflection in the y-axis.

Summary of Section 2

This section has introduced:

◇   the idea of finding information about the solution to a problem by using the graph of an appropriate function;

◇   a technique for sketching the graph of a quadratic function using the completed-square form;

◇   techniques for translating and scaling a known graph $y = f(x)$.

Graph	Translation or scaling of $y = f(x)$
$y = f(x + p)$	Horizontal translation by p units to the left (right if p is negative)
$y = f(x) + q$	Vertical translation by q units upwards (downwards if q is negative)
$y = af(x)$	y-scaling with factor a
$y = f(bx)$	x-scaling with factor $1/b$

Exercises for Section 2

Exercise 2.1

A square garden with side length 8 metres is to have uniform borders around *three* sides, the rest being lawn. The borders are to take half the area of the garden. Let x denote the width of the border (in metres), and let A denote the area of the lawn (in square metres).

Solve this problem by determining

(a) the largest closed interval consisting of values of x which make sense for this problem;

(b) the area A of the lawn, in terms of x;

(c) the value of x for which A is half the area of the garden.

Exercise 2.2

(a) Sketch the graph of the function

$$f(x) = 2x^2 - 24x + 64,$$

by using translations and scalings of the graph of $y = x^2$, and by finding the x- and y-intercepts.

(b) What is the relevance of the graph in part (a) to the 'garden problem' in Exercise 2.1?

Exercise 2.3

Sketch the graph of the function

$$f(x) = -\frac{2}{x + 1} + 1,$$

by using translations and scalings of the graph of $y = 1/x$, and by finding the x- and y-intercepts.

3 Trigonometric and exponential functions

In this section, two important classes of functions are introduced, both based upon ideas that you have seen before. *Trigonometric functions* follow from the introduction of the sine, cosine and tangent of an angle, and *exponential functions* relate to powers of numbers and to geometric sequences. Both of these classes of functions have very significant applications within mathematical models of various real-life phenomena. Examples include the oscillating level of a tide (sine and cosine functions) and radioactive decay (exponential functions).

See Chapter A1, Section 3.

3.1 Trigonometric functions

We start by recalling the definitions of *cosine* and *sine*. If $P(x, y)$ is any point on the unit circle, as shown in Figure 3.1, and the line segment OP is at an angle t measured from the positive x-axis, then

$$\cos t = x \quad \text{and} \quad \sin t = y.$$

Here the angle t may be of any size (positive, negative or zero); that is, t may take any real value.

Recall that the unit circle has centre at the origin O and radius 1.

Here, and later in this chapter, it is assumed that t is measured in radians. Recall that π radians $= 180°$. The symbol θ rather than t was used in Chapter A2, Subsection 3.1.
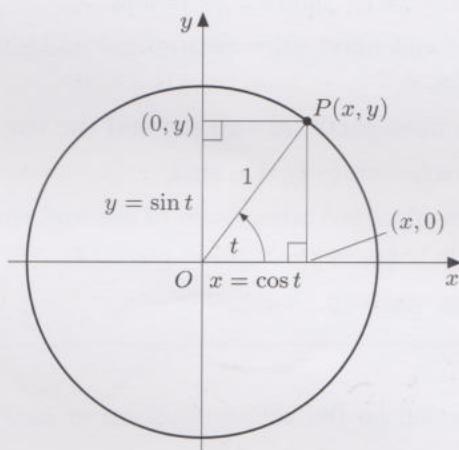
Figure 3.1 Defining cos and sin

Now cos and sin, as defined above, are *functions* in the sense described in Section 1. There is no simple formula to calculate the values of these **cosine** and **sine** functions, but the geometric definitions of $\cos t$ and $\sin t$ provide the rules for a pair of functions nonetheless. For each input value t, there are unique output values $x = \cos t$ and $y = \sin t$. Since t may take any real value,

the domain of each of the functions cos and sin is $\mathbb{R}$.

In the next activity you are asked to describe what image values can be obtained from these two functions.

Activity 3.1   Set of images under cos and sin

By referring to the definitions of cos and sin in terms of the unit circle, write down the image set – that is, the complete set of image values – for each of these functions.

Solutions are given on page 52.

We seek next to sketch the graphs of the trigonometric functions cos and sin. A first step might be, as with the functions considered previously, to draw up a table of values. A calculator could be used to obtain values for a table directly, but in the absence of a simple formula for the rules of cos and sin, these cannot be checked independently except in a few special cases. We look at these special cases first.

See also Chapter A2, Subsection 3.1.

Activity 3.2   Images of cos and sin in special cases

(a) By referring to the unit circle or to an appropriate right-angled triangle in each case, find the values obtained when each of cos and sin is applied to the inputs 0, $\frac{1}{6}\pi$, $\frac{1}{4}\pi$, $\frac{1}{3}\pi$, $\frac{1}{2}\pi$.

You may like to check that your calculator gives the same values with these inputs. Remember that your calculator should be in radian mode.

(b) Use your results from part (a) and the trigonometric formulas
$$\cos(\pi - t) = -\cos t, \quad \sin(\pi - t) = \sin t$$
to find the values obtained when each of cos and sin is applied to the inputs $\frac{2}{3}\pi$, $\frac{3}{4}\pi$, $\frac{5}{6}\pi$, π.

You met the trigonometric formulas in parts (b) and (c) in Chapter A2, Subsection 3.1. If you are confident about finding values of trigonometric functions, then you may prefer to use an alternative method in parts (b) and (c).

(c) Use your results from parts (a) and (b) and the trigonometric formulas
$$\cos(-t) = \cos t, \quad \sin(-t) = -\sin t$$
to find the values obtained when each of cos and sin is applied to the inputs $-\frac{1}{6}\pi$, $-\frac{1}{4}\pi$, $-\frac{1}{3}\pi$, $-\frac{1}{2}\pi$, $-\frac{2}{3}\pi$, $-\frac{3}{4}\pi$, $-\frac{5}{6}\pi$, $-\pi$.

Solutions are given on page 52.

The table below is based on the values obtained in Activity 3.2.

The values of $\cos t$ and $\sin t$ in the table are given correct to two decimal places.

t	$-\pi$	$-\frac{5}{6}\pi$	$-\frac{3}{4}\pi$	$-\frac{2}{3}\pi$	$-\frac{1}{2}\pi$	$-\frac{1}{3}\pi$	$-\frac{1}{4}\pi$	$-\frac{1}{6}\pi$	0
$\cos t$	-1	-0.87	-0.71	-0.5	0	0.5	0.71	0.87	1
$\sin t$	0	-0.5	-0.71	-0.87	-1	-0.87	-0.71	-0.5	0

t	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
$\cos t$	1	0.87	0.71	0.5	0	-0.5	-0.71	-0.87	-1
$\sin t$	0	0.5	0.71	0.87	1	0.87	0.71	0.5	0

This table covers a selection of values for t within the interval $[-\pi, \pi]$. Outside this range, we can apply the formulas

See Chapter A2, Subsection 3.1.

$$\cos(t + 2\pi) = \cos t, \quad \sin(t + 2\pi) = \sin t,$$

which describe the fact that each of the graphs of the functions cos and sin repeats itself after every 2π radians; see Figure 3.2 below. We say that a function f is **periodic**, with **period** p, if $f(t + p) = f(t)$ for all t in the domain of f. Thus the cosine and sine functions are both periodic, with period 2π.
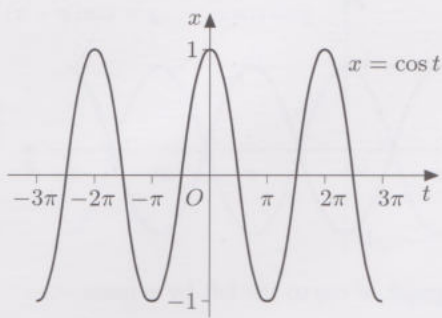
(a) Graph of cos

(b) Graph of sin

Figure 3.2   cos and sin are periodic

We have put the graphs of cos and sin on separate sets of axes, since the *dependent* variables are different: x for the cosine function and y for the sine function. However, if we concentrate on cos and sin as functions, and leave in the background their definitions in terms of the unit circle, then both can be expressed in the form $y = f(x)$, with x as the independent variable and y as the dependent variable. With this choice of variables, both graphs can be placed on the same set of axes, as shown in Figure 3.3 below.

Remember that a function is independent of the variables which are chosen to label its inputs and outputs and to display its graph.
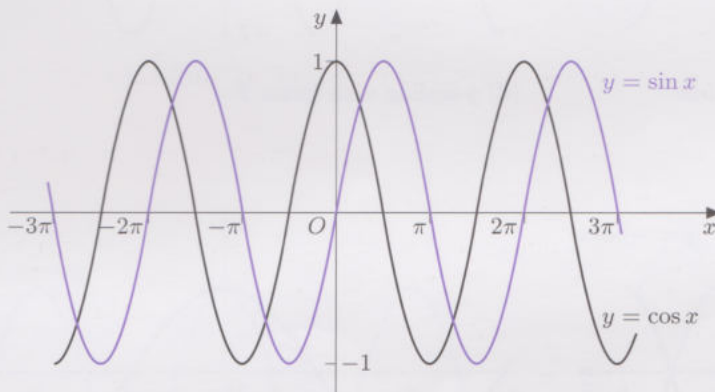
Figure 3.3   Graphs of cos and sin on the same axes

Function notation would lead us to write $\cos(x)$ and $\sin(x)$, but where there is no ambiguity we often omit the brackets.

As with the quadratic functions in Section 2, it is possible to *translate* or *scale* these graphs of cos and sin in either the x- or the y-direction; see Figure 3.2. For example, the graph of the function $f(x) = \cos(x + p)$ is obtained by translating the graph of $\cos x$ by p units to the left. This and other transformations of the graph of cos are illustrated in Figure 3.4 overleaf.

Similar effects can be demonstrated starting from the graph of the sine function.
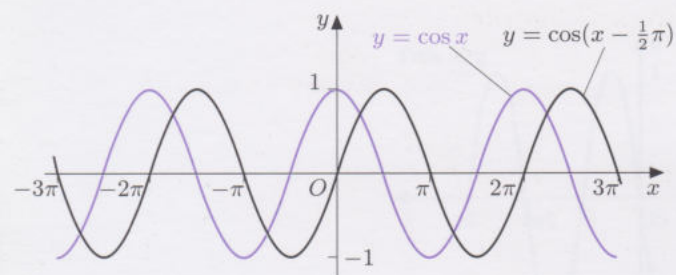
(a) translation of graph of cos to the right by $\frac{1}{2}\pi$ units
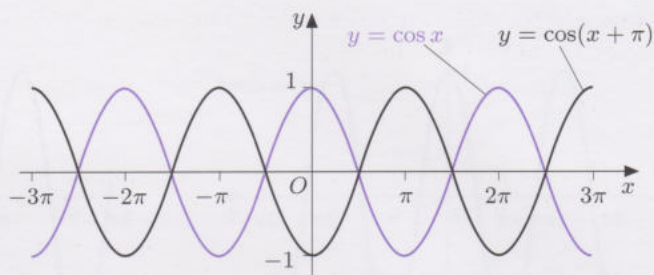
(b) translation of graph of cos to the left by π units
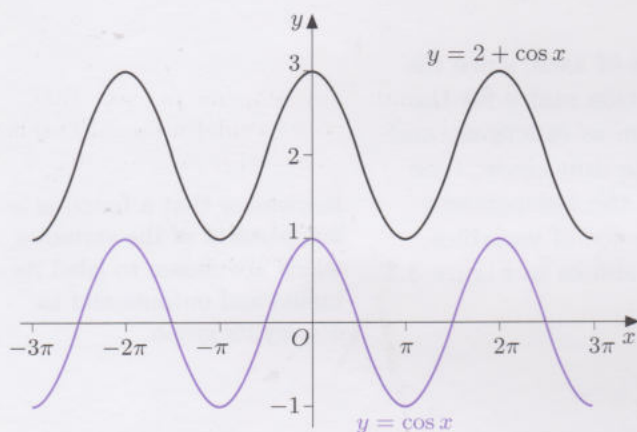
(c) translation of graph of cos by 2 units upwards
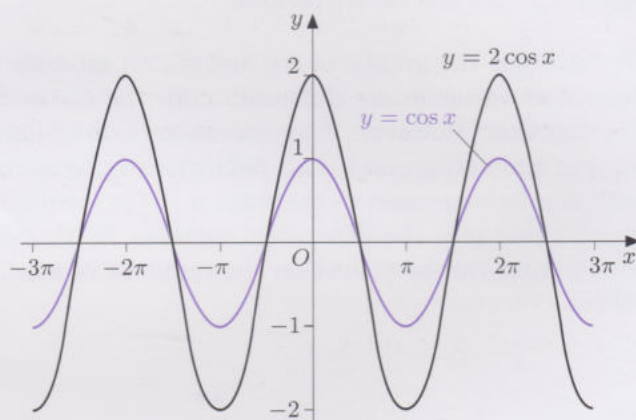
(d) y-scaling with factor 2
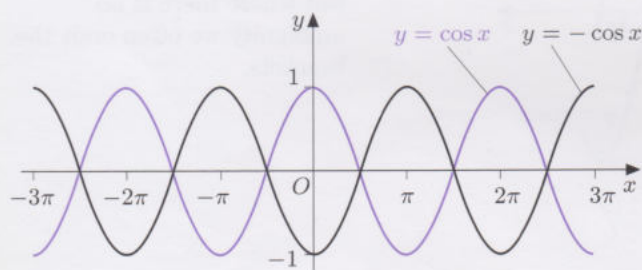
(e) y-scaling with factor -1
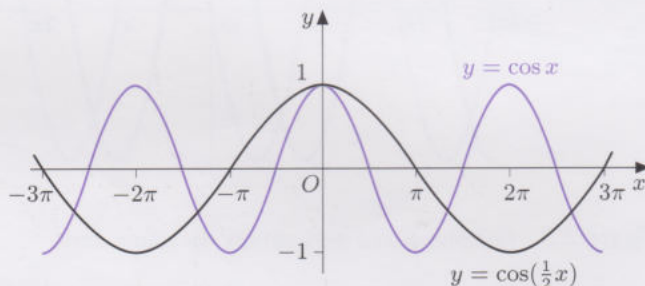
(f) x-scaling with factor 2

Figure 3.4   Transforming the graph of cos

Note that the graph of $y = \cos(x - \frac{1}{2}\pi)$ in Figure 3.4(a) is the same as the graph of $y = \sin x$. This reflects the fact that, for all values of x, we have $\cos(x - \frac{1}{2}\pi) = \sin x$. Similarly, the graphs in Figures 3.4(b) and 3.4(e) are identical, since $\cos(x + \pi) = -\cos x$ for all x.

Activity 3.3   Transforming the graph of cos

Graphs that can be obtained from that of the sine function (or cosine function) by translation and/or scaling are often called 'sinusoidal'.

By considering the effect of an appropriate translation or scaling on the graph of cos, sketch the graph of each of the following functions.

(a) $y = -3 + \cos x$ (b) $y = \frac{1}{2}\cos x$ (c) $y = \cos(3x)$ (d) $y = \cos(-x)$

Solutions are given on page 53.

This subsection concludes by considering briefly the **tangent** function. This is defined by the rule

$$\tan x = \frac{\sin x}{\cos x},$$

which produces an output for all real numbers x *except* when $\cos x = 0$. Figure 3.3 shows that $\cos x = 0$ occurs for each of the values $x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \ldots$. Hence the domain of tan is all of $\mathbb{R}$ excluding these values, so it is composed of the open intervals

$$\ldots, \; (-\tfrac{5}{2}\pi, -\tfrac{3}{2}\pi), \; (-\tfrac{3}{2}\pi, -\tfrac{1}{2}\pi), \; (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi), \; (\tfrac{1}{2}\pi, \tfrac{3}{2}\pi), \; (\tfrac{3}{2}\pi, \tfrac{5}{2}\pi), \; \ldots.$$

Let us concentrate first on the central interval, $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

Activity 3.4   Images of tan in special cases

(a) By using the definition of tan and your answers to Activity 3.2(a), find the values obtained when tan is applied to the inputs $0, \frac{1}{6}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi$.

(b) Use your results from part (a) and the trigonometric formula

$$\tan(-x) = -\tan x$$

to find the values obtained when tan is applied to the inputs $-\frac{1}{6}\pi, -\frac{1}{4}\pi, -\frac{1}{3}\pi$.

Solutions are given on page 53.

You may like to check that your calculator gives the same values with these inputs.

We could now draw up a table of values for the function tan within the central interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. However, to cut a long story short, the graph of tan is as shown in Figure 3.5.
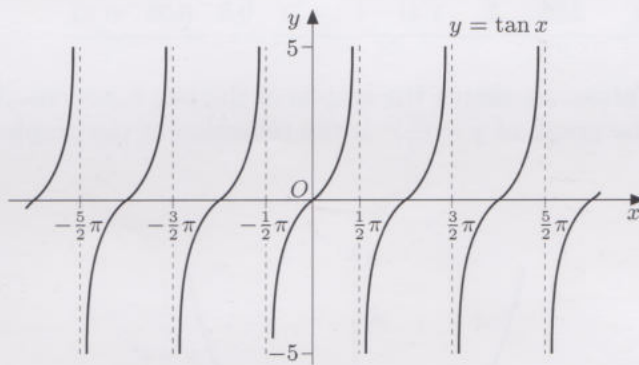
Figure 3.5   Graph of tan

The behaviour of $\tan x$ within the central interval as x approaches $\frac{1}{2}\pi$ (from the left) can be deduced from the rule $\tan x = \sin x / \cos x$, by noting that $\sin x$ is close to $\sin(\frac{1}{2}\pi) = 1$ whereas $\cos x$ approaches zero through positive values. Similarly, as x approaches $-\frac{1}{2}\pi$ from the right, $\sin x$ is close to $\sin(-\frac{1}{2}\pi) = -1$ while $\cos x$ again approaches zero through positive values. So as x approaches $\frac{1}{2}\pi$ from the left, $\tan x$ becomes large through positive values. Similarly, as x approaches $-\frac{1}{2}\pi$ from the right, $\tan x$ becomes large through negative values.

It can be seen from the graph that the image set of tan is $\mathbb{R}$.

Outside the central interval, the graph is determined by the fact that the function tan is periodic with period π; that is, $\tan(x + \pi) = \tan x$. Translating the graph of tan horizontally by any integer multiple of π (either to left or right) gives the same graph once more. For this graph, each of the vertical lines $x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \ldots$ is an asymptote.

This fact holds because

$$\frac{\sin(x + \pi)}{\cos(x + \pi)} = \frac{-\sin x}{-\cos x};$$

see Chapter A2, Exercise 3.2(a).

3.2 Exponential functions

If a is a *positive* real number and n is a positive integer, then a^n represents a multiplied by itself n times. A meaning can also be assigned in a natural way to a^x, where x is any real number. When x is not a positive integer, this is achieved as follows.

See Chapter A0, Subsection 3.1.

Here $x = 0$.

Here $x = -n$.

Here $x = p/q$.

(i) $a^0 = 1$.

(ii) If n is a positive integer, then $a^{-n} = 1/a^n$.

(iii) If p, q are integers, with $q > 0$, then $a^{p/q} = (\sqrt[q]{a})^p$.

(iv) If x is an irrational number, then the value of a^x can be found to any desired accuracy by approximating x more and more closely by rational numbers p/q, for which $a^{p/q}$ is defined by (iii) above.

With these definitions, we have the following rules for powers, where x and y are any real numbers:

$$a^{x+y} = a^x a^y \quad \text{and} \quad (a^x)^y = a^{xy}. \tag{3.1}$$

For each positive real number a, the process just described for finding values of a^x for any real number x is a rule for the function

There is one such function for each positive real number a.

$$f(x) = a^x,$$

with domain $\mathbb{R}$. Each such function is called an **exponential function**, and the number a is called the **base** of the function.

We can draw up tables of values for such functions. Below is a table for the exponential functions with base 2 and base $\frac{1}{2}$, for values of x within the interval $[-2, 2]$.

Note that $(\frac{1}{2})^x$ can also be written as 2^{-x} or as $1/2^x$.

The values of 2^x and $(\frac{1}{2})^x$ in the table are given correct to two decimal places.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
2^x	0.25	0.35	0.5	0.71	1	1.41	2	2.83	4
$(\frac{1}{2})^x$	4	2.83	2	1.41	1	0.71	0.5	0.35	0.25

Using these values, we sketch the graphs of the two functions. Because $(\frac{1}{2})^x = 2^{-x}$, the graph of $y = (\frac{1}{2})^x$ is the reflection of the graph of $y = 2^x$ in the y-axis.
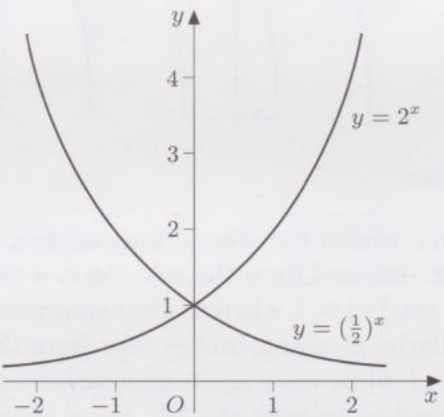

Figure 3.6 Graphs of two exponential functions

Activity 3.5 Sketching graphs of exponential functions

(a) Use your calculator, where necessary, to find image values of the function $f(x) = 5^x$ for $x = -1, -0.75, -0.5, \ldots, 1$.

(b) Use your results from part (a) to sketch the graph of the function $f(x) = 5^x$ for x in $[-1, 1]$.

(c) Explain how, without performing any further calculations, you could use the result from part (b) to sketch the graph of the function $g(x) = \left(\frac{1}{5}\right)^x$.

Solutions are given on page 53.

These graphs may bring to mind the graphs which arise from geometric sequences. The closed forms of geometric sequences with initial term 1 can be written as $y_n = a^n$ $(n = 0, 1, 2, \ldots)$.

See Chapter A1, Section 3.

You may recall from the study of geometric sequences that:

(i) if $a > 1$, then $a^n \to \infty$ as $n \to \infty$;

(ii) if $0 < a < 1$, then $a^n \to 0$ as $n \to \infty$.

Similar properties hold for the exponential functions:

(i) if $a > 1$, then $a^x \to \infty$ as $x \to \infty$;

(ii) if $0 < a < 1$, then $a^x \to 0$ as $x \to \infty$.

The graphs of geometric sequences in Chapter A1 consisted of isolated points, whereas the graphs of exponential functions are smooth curves.

These forms of behaviour for large positive values of x are evident in the graphs of the two exponential functions sketched in Figure 3.6. In general, the graph of $f(x) = a^x$, where $a > 0$, has one of the three forms shown in Figure 3.7 below.

(a) $a > 1$ (b) $a = 1$ (c) $0 < a < 1$

The image set of any exponential function $f(x) = a^x$, for $a > 1$ or $0 < a < 1$, is the open interval $(0, \infty)$. Also, the x-axis is an asymptote of the graph of f.
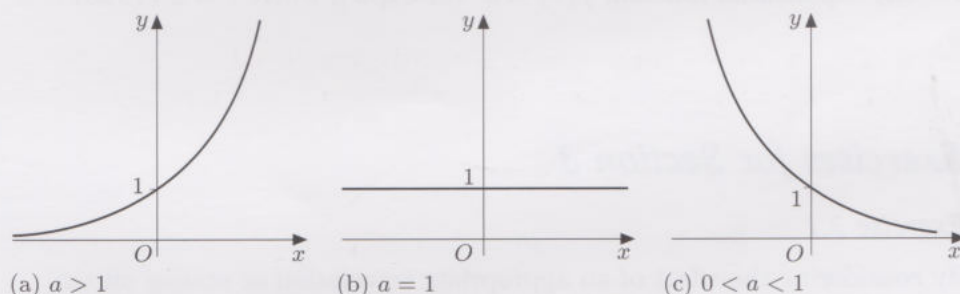
Figure 3.7   Graphs of exponential functions $f(x) = a^x$: three cases

While an exponential function of the form $f(x) = a^x$ is defined for each positive number a, one value of a is worthy of particular mention. This is the number $e = 2.718\,281\ldots$. Like π, the number e is an irrational number which can be determined to as many decimal places as desired. There are various equivalent ways of defining e. One of these is that e is the unique number such that the graph of $y = e^x$ has a tangent line at $(0, 1)$ whose slope is 1; see Figure 3.8, overleaf.

Why this feature is singled out to define e will become more apparent when you study the calculus in Block C. In that block the notion of a tangent line to a curve is discussed.
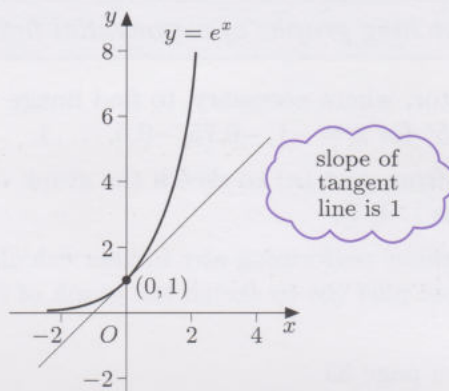
Figure 3.8   Graph of the function $f(x) = e^x$

The function $f(x) = e^x$ is often referred to as *the* exponential function, and is denoted also by exp. Thus $\exp x = e^x$ and $\exp(-x) = e^{-x}$.

Your calculator should have an e^x or exp button, which calculates values of this exponential function. In particular, calculating $e^1 = \exp 1$ will give you an approximate value for e.

Summary of Section 3

This section has introduced:

◇   the trigonometric functions cos, sin and tan, together with their graphs;

◇   periodic functions;

◇   the exponential functions $f(x) = a^x$ (where $a > 0$), together with their graphs;

◇   the exponential function $f(x) = e^x$ (or $\exp x$), where $e = 2.718\,281\ldots$.

Exercises for Section 3

Exercise 3.1

By considering the effect of an appropriate translation or scaling on the graph of sin, sketch each of the following graphs.

(a) $y = 1 + \sin x$ (b) $y = -\sin x$ (c) $y = \sin(4x)$

Exercise 3.2

By considering the effect of an appropriate translation on the graph of the function $f(x) = e^x$ (see Figure 3.8), sketch the graph of the function $g(x) = e^{x+1}$.

4 Inverse functions

In Section 1, functions were introduced as a way to study a particular type of relationship between a dependent variable and an independent variable. Often, however, we wish to reverse the roles of dependent and independent variables. For example, the formula $C = 2\pi r$ for the circumference C of a circle of radius r can be rearranged as

$$r = \frac{1}{2\pi} C,$$

to express the radius in terms of the circumference. This idea leads to the concept of an *inverse function*.

4.1 What is an inverse function?

Consider the two 'squaring functions'

$$g(x) = x^2 \ (x \text{ in } [0, \infty)) \quad \text{and} \quad h(x) = x^2. \tag{4.1}$$

These functions have the same rule, but the domain of g is $[0, \infty)$, whereas the domain of h is $\mathbb{R}$, by the domain convention; see Figure 4.1.

(a) Graph of $g(x) = x^2$ (x in $[0, \infty)$)   (b) Graph of $h(x) = x^2$
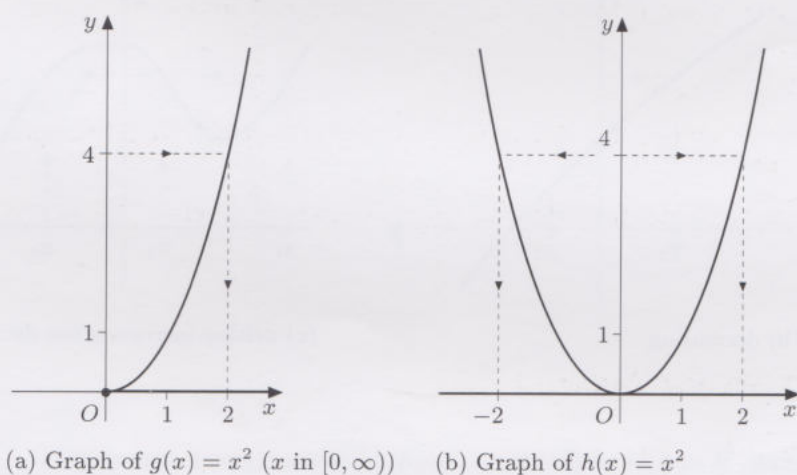
Figure 4.1   Two squaring functions

The functions g and h have the same image set, namely $[0, \infty)$.

If we now consider trying to reverse the effects of g and h, then we find a significant difference between these two functions:

Recall that the image set of a function is the set of all possible image values of the function.

◇   for each value of y in the image set $[0, \infty)$, there is *exactly one* value of x in $[0, \infty)$ such that $g(x) = y$;

◇   for each value of y in the image set $[0, \infty)$, apart from 0, there are *two* values of x in $\mathbb{R}$ such that $h(x) = y$.

This difference is illustrated in Figure 4.1, with the value $y = 4$.

A consequence of this difference is that it is not possible to find a function which reverses the effect of h. As you will see, however, the effect of g *can* be reversed by a function.

Recall that a function has to give a *unique* output for each input.

To describe this difference, some new ideas are introduced. A function f is said to be **one-one**, or **one-to-one**, if it has the following property:

for all x_1, x_2 in the domain of f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

A function which is not one-one is said to be **many-one**, or **many-to-one**.

The function h in Figure 4.1(b) is many-one; for example, $h(-2) = h(2)$.

In terms of the graph of a function, being one-one means that each horizontal line which meets the graph does so *exactly once*. The function g defined in equations (4.1) has this property since, as can be seen in Figure 4.1(a), this function is *increasing*. As you take greater values of x (further to the right on the x-axis), the corresponding values of $g(x)$ are greater (further up the y-axis). Written in symbols, a function f is said to be **increasing** if it has the following property:

In some texts, functions with these properties are called 'strictly increasing' and 'strictly decreasing'. Note also that these definitions provide formal meanings for everyday words.

for all x_1, x_2 in the domain of f, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.

Similarly, a function f is **decreasing** if it has the following property:
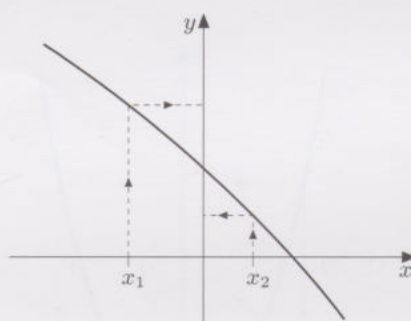
for all x_1, x_2 in the domain of f, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

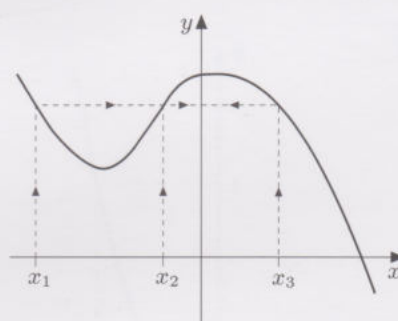If a function is either increasing or decreasing, then it is certainly one-one.

For example, in Figure 4.2, the function on the left is increasing (so it is one-one), the function in the middle is decreasing (so it is one-one), but the function on the right is neither increasing nor decreasing (it is many-one).
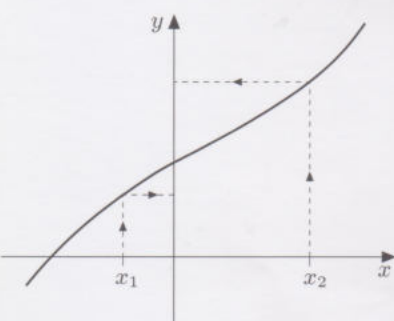
(a) increasing   (b) decreasing   (c) neither increasing nor decreasing

Figure 4.2   Types of behaviour

Sometimes, it can be difficult to check whether a given function is increasing or decreasing, but in this chapter we consider only functions where this can be deduced immediately from the graph of the function.

Activity 4.1   Increasing functions and decreasing functions

For each of the following functions, state whether the function is: increasing, decreasing, neither increasing nor decreasing, one-one, many-one.

(a) $f(x) = \cos x$ (see Figure 3.3)

(b) $g(x) = \sin x$ $(-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi)$ (see Figure 3.3)

(c) $h(x) = (\frac{1}{2})^x$ (see Figure 3.6)

Solutions are given on page 54.

For any one-one function f, it is possible to define an **inverse function**, or **inverse** for short, denoted by f^{-1}. As shown in Figure 4.3, the domain of the inverse function f^{-1} is the image set of f, the set consisting of all possible image values $f(x)$, and the rule of the inverse function is

$$f^{-1}(y) = x, \quad \text{where} \quad y = f(x).$$

This inverse function f^{-1} reverses, or undoes, the effect of f.

Also, f undoes the the effect of f^{-1}, which is itself one-one.

For example, the function $h(x) = (\frac{1}{2})^x$ is one-one (see Activity 4.1(c)), so it has an inverse function h^{-1}, which undoes the effect of h. In particular:

$$h(1) = \tfrac{1}{2}, \quad \text{so} \quad h^{-1}(\tfrac{1}{2}) = 1;$$
$$h(2) = \tfrac{1}{4}, \quad \text{so} \quad h^{-1}(\tfrac{1}{4}) = 2.$$

The notation f^{-1} should not be confused with $1/f$, which is the function with rule $x \longmapsto 1/f(x)$. Shortly, you will see that particular inverse functions have special names.
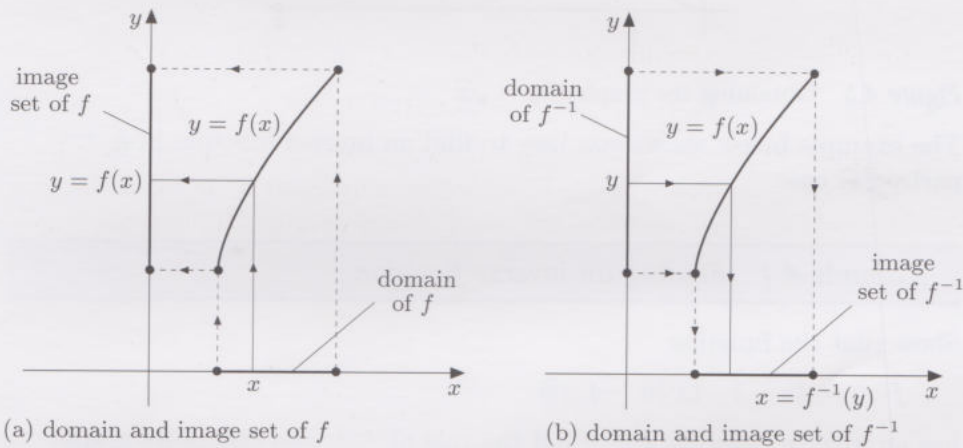

(a) domain and image set of f

(b) domain and image set of f^{-1}

Figure 4.3   A function and its inverse

For convenience, the domain of f is taken to be a closed interval in this figure.

In Figure 4.3(b) you can also see that the image set of the inverse function f^{-1} is equal to the domain of f. Furthermore, as shown in Figure 4.4, if f^{-1} is written as a function of x, in the usual way, then the graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by exchanging the roles of x and y. This exchange of roles corresponds to reflecting the graph of $y = f(x)$ in the 45° line, as indicated by the double-headed arrow, and using the old x-scale on the new y-axis and vice versa.

In the case of equal scales on the axes, the 45° line is the line $y = x$, as in Figure 4.5.

(a) Graph of $y = f(x)$
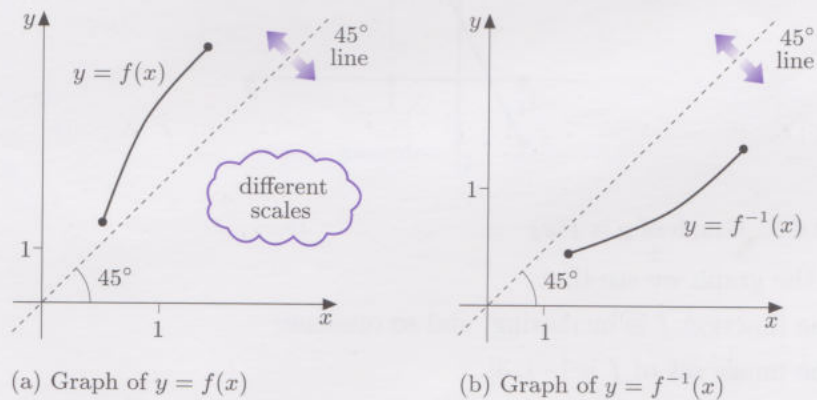
(b) Graph of $y = f^{-1}(x)$

Figure 4.4   Obtaining the graph of $y = f^{-1}(x)$ by reflection

For example, the function $g(x) = x^2$ (x in $[0, \infty)$), considered earlier, is one-one, and its inverse function g^{-1} is given by

$$g^{-1}(y) = \sqrt{y} \quad (y \text{ in } [0, \infty)).$$

Exchanging the roles of x and y, the graph of $y = g^{-1}(x) = \sqrt{x}$ is the reflection of the graph of g in the 45° line; see Figure 4.5, in which the x-scale and the y-scale are the same.
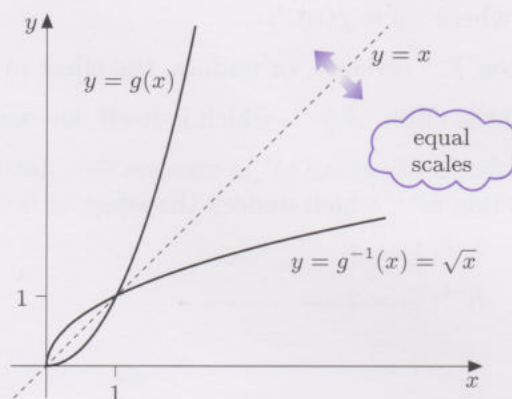

Figure 4.5   Obtaining the graph of $y = \sqrt{x}$

The example below shows you how to find an inverse function in a particular case.

Example 4.1   Finding an inverse function

Show that the function

$$f(x) = 2x + 1 \quad (x \text{ in } [-1, 1])$$

has an inverse function f^{-1}. Find the rule of f^{-1}, and sketch its graph.

Solution

First we sketch the graph of f. This is the graph of $y = 2x + 1$, restricted to the closed interval $[-1, 1]$.
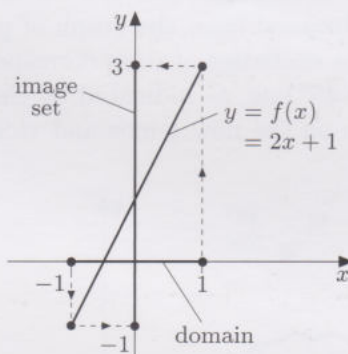

Figure 4.6   Graph of $y = f(x)$

From the graph we see that:

◇   the function f is increasing, and so one-one;

◇   the image set of f is $[-1, 3]$.

Therefore f has an inverse function f^{-1} with domain $[-1, 3]$ and image set $[-1, 1]$. We can find the rule of f^{-1} by solving the equation

$$y = f(x) = 2x + 1$$

to obtain x in terms of y:

$$y = 2x + 1, \quad \text{so} \quad x = \tfrac{1}{2}(y - 1).$$

If the domain of f had been the *open* interval $(-1, 1)$, then the image set would have been $(-1, 3)$.

Thus the inverse function is

$$f^{-1}(y) = \tfrac{1}{2}(y-1) \quad (y \text{ in } [-1,3]),$$

which, expressed in terms of x, is

$$f^{-1}(x) = \tfrac{1}{2}(x-1) \quad (x \text{ in } [-1,3]).$$

The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ in the 45° line; see Figure 4.7.
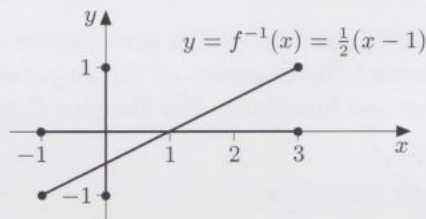

Figure 4.7   Graph of $y = f^{-1}(x)$

In the following activity, you have the opportunity to practise the above method of finding inverse functions. In part (c) of this activity, the graph of f requires different scales on the axes. So, when reflecting the graph of f in the 45° line to obtain the graph of f^{-1}, remember to exchange the scales as well.

Activity 4.2   Finding inverse functions

For each of the following functions f, find the inverse function f^{-1} and sketch the graph of $y = f^{-1}(x)$.

(a) $f(x) = 3x - 1 \quad (x \text{ in } (0,2))$

(b) $f(x) = x^3 \quad (x \text{ in } [0,\infty))$

(c) $f(x) = 4x^2 - 56x + 192 \quad (x \text{ in } [0,6])$

   (*Hint*: In this part, make use of the solution to Activity 2.5. Also, when solving the equation $y = f(x)$ to obtain x in terms of y, consider carefully which solution to choose.)

Solutions are given on page 54.

Comment

Note that we can use the inverse function found in part (c) to solve the exhibition hall problem. Since the function

$$f(x) = 4x^2 - 56x + 192 \quad (x \text{ in } [0,6])$$

has inverse function

$$f^{-1}(y) = 7 - \sqrt{1 + \tfrac{1}{4}y} \quad (y \text{ in } [0,192]),$$

the solution of the equation $f(x) = 96$ which lies in $[0,6]$ is

$$x = f^{-1}(96) = 7 - \sqrt{1 + \tfrac{1}{4} \times 96} = 7 - \sqrt{25} = 2,$$

as expected.

See Subsection 2.1.

4.2 Inverse trigonometric functions

When working with trigonometry, we often wish to find an angle θ for which we know the value of $\sin\theta$, $\cos\theta$ or $\tan\theta$. It is always necessary to be careful when finding such a θ, since equations of the form

$$\sin\theta = k, \quad \cos\theta = k \quad \text{and} \quad \tan\theta = k \tag{4.2}$$

do not have unique solutions. For example, the equation $\sin\theta = 0$ has infinitely many solutions: $\theta = 0, \pm\pi, \pm 2\pi, \ldots$. In particular, the sine function is not one-one (and nor are the cosine and tangent functions).

In order to define inverse functions which give unique solutions to equations (4.2), we restrict the domains of the sine, cosine and tangent functions to obtain one-one functions. For the sine function, we restrict the domain as follows:

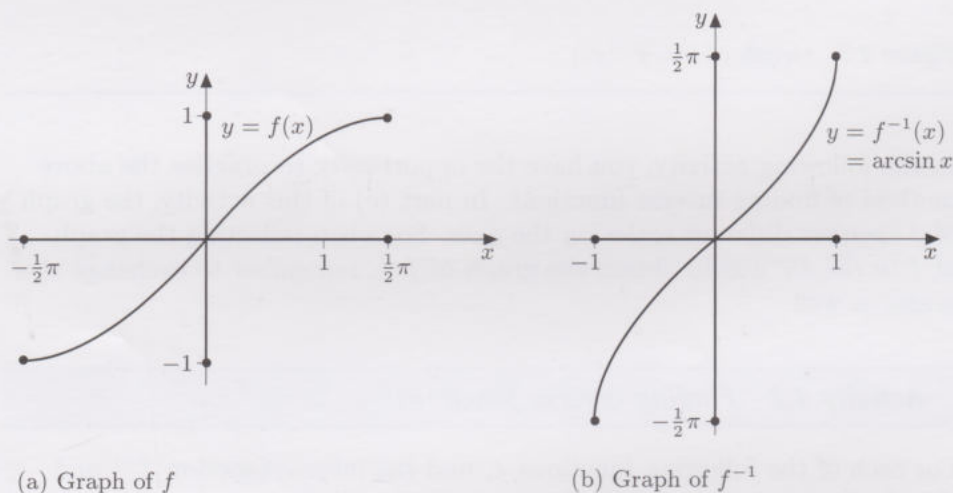$$f(x) = \sin x \quad (-\tfrac{1}{2}\pi \le x \le \tfrac{1}{2}\pi).$$

(a) Graph of f

(b) Graph of f^{-1}

Figure 4.8   Graph of $f(x) = \sin x$ $(-\tfrac{1}{2}\pi \le x \le \tfrac{1}{2}\pi)$ and its inverse f^{-1}

Figure 4.8(a) shows that:

◇ the function f is increasing, and so one-one;

◇ the image set of f is $[-1, 1]$.

Therefore f has an inverse function f^{-1} with domain $[-1, 1]$ and image set $[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]$. The graph of f^{-1}, obtained by reflecting the graph of f in the $45°$ line, is shown in Figure 4.8(b).

The name 'arcsine' arises from an old meaning of 'arc' as 'angle'.

This inverse function is given the name **arcsine**. Thus, for $-1 \le y \le 1$,

$$x = \arcsin y \quad \text{means that} \quad y = \sin x \text{ and } -\tfrac{1}{2}\pi \le x \le \tfrac{1}{2}\pi.$$

In words, for $-1 \le y \le 1$,

$\arcsin y$ is that angle in $[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]$ whose sine is y.

We can calculate some particular values of the arcsine function, by using our knowledge of the sine function. For example, the value of $\arcsin 0$ is that angle in $[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]$ whose sine is 0. We know that this angle is 0, so

$$\arcsin 0 = 0.$$

Similarly, since

$$\sin(\tfrac{1}{6}\pi) = \tfrac{1}{2} \quad \text{and} \quad \sin(-\tfrac{1}{2}\pi) = -1,$$

we have

$$\arcsin(\tfrac{1}{2}) = \tfrac{1}{6}\pi \quad \text{and} \quad \arcsin(-1) = -\tfrac{1}{2}\pi.$$

In a similar way, we can restrict the domains of the cosine and tangent functions to define the inverse functions **arccosine** and **arctangent**. The details are given in Figures 4.9 and 4.10.

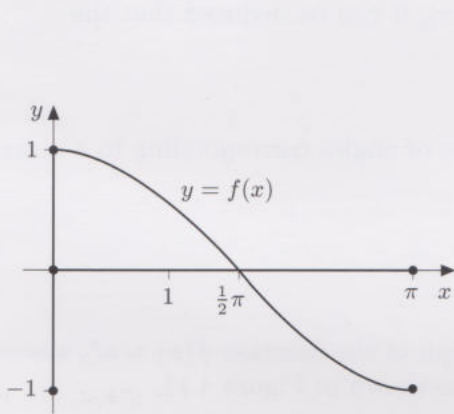Alternative names for the inverse trigonometric functions are

$$\sin^{-1},\ \cos^{-1},\ \tan^{-1}.$$
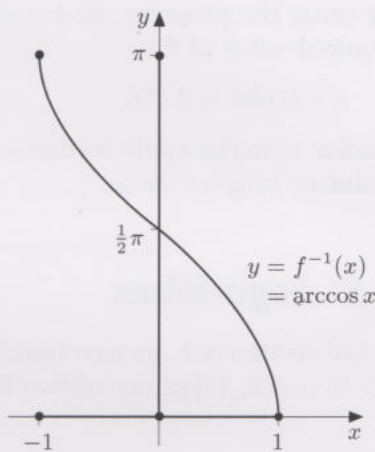
Calculators and computers may also use the names

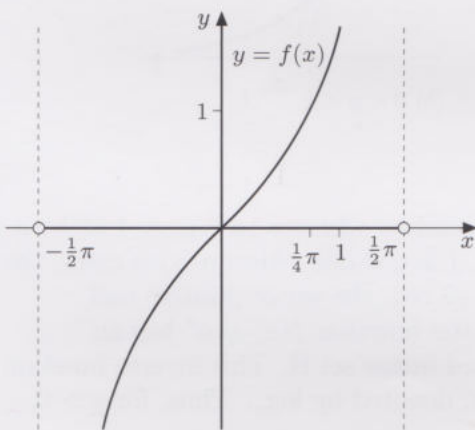INV SIN, INV COS, INV TAN

or
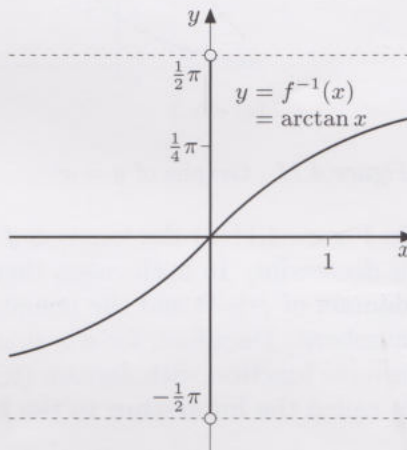
asin, acos, atan.

(a) Graph of f (b) Graph of f^{-1}

Figure 4.9   Graph of $f(x) = \cos x$ $(0 \le x \le \pi)$ and its inverse f^{-1}

(a) Graph of f (b) Graph of f^{-1}

Figure 4.10   Graph of $f(x) = \tan x$ $(-\frac{1}{2}\pi < x < \frac{1}{2}\pi)$ and its inverse f^{-1}

Notice that the graph of arctan has two horizontal asymptotes:

$$y = \pm \tfrac{1}{2}\pi.$$

Activity 4.3   Evaluating inverse trigonometric functions

(a) Which of the following are valid expressions?
  (i) $\arcsin(2\pi)$ (ii) $\arccos 0$ (iii) $\arctan(\frac{1}{2}\pi)$
  (iv) $\arctan(\tan(\frac{5}{4}\pi))$

(b) Write down exact values for each of the following expressions, giving angles in radians.
  (i) $\arcsin(\frac{1}{2}\sqrt{3})$ (ii) $\arccos(-0.5)$ (iii) $\arctan 1$
  (iv) $\arcsin(\sin(\frac{1}{5}\pi))$ (v) $\cos(\arccos(0.9))$
  (vi) $\arctan(\tan(\frac{5}{4}\pi))$

You should not need to use your calculator in this activity.

Solutions are given on page 56.

Finding a value of θ

If you know that $\sin\theta = 0.61$, for example, then the function arcsin gives a value of θ in the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ whose sine is 0.61, namely 0.656 (to 3 d.p.). But you may also know that θ lies in the interval $[\frac{1}{2}\pi, \pi]$. By using the property $\sin x = \sin(\pi - x)$, it can be deduced that the required value of θ is

$$\pi - 0.656 \simeq 2.486.$$

Similar remarks apply to finding values of angles corresponding to a given cosine or tangent value.

4.3 Logarithms

In Subsection 3.2, we saw that the graph of the function $f(x) = a^x$, where $a > 0$, $a \neq 1$, takes one of two forms, as shown in Figure 4.11.
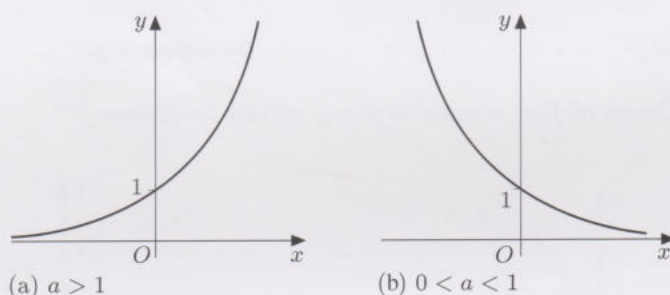

(a) $a > 1$ (b) $0 < a < 1$

Figure 4.11 Graphs of $y = a^x$

The function $g(x) = 1^x = 1$ is a constant function, which is neither increasing nor decreasing.

In Figure 4.11(a) the function f is increasing, whereas in Figure 4.11(b) it is decreasing. In both cases, therefore, f is one-one. Also in both cases, the domain of f is $\mathbb{R}$ and the image set is $(0, \infty)$, the set of positive real numbers. Therefore, for $a > 0$, $a \neq 1$, the function $f(x) = a^x$ has an inverse function with domain $(0, \infty)$ and image set $\mathbb{R}$. This inverse function is called the **logarithm** to the **base** a, denoted by $\log_a$. Thus, for $y > 0$,

We read $\log_a y$ as 'log to the base a of y'.

$$x = \log_a y \quad \text{means that} \quad y = a^x;$$

in words,

the logarithm to the base a of y is that power of a which equals y.

Note that $x = \log_a(a^x)$, for x in $\mathbb{R}$, and $y = a^{\log_a y}$, for $y > 0$.

For example, the value of $\log_2 8$ is that power of 2 which equals 8. Since $8 = 2^3$, we have
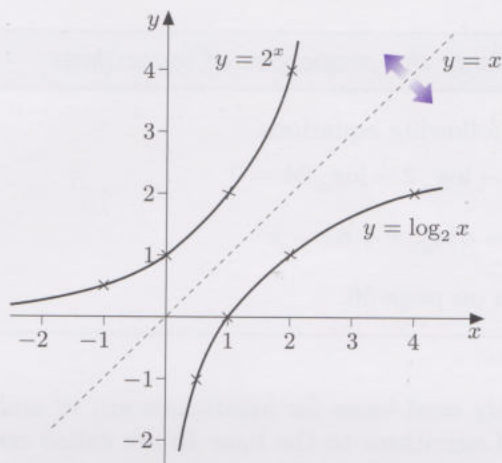
$$\log_2 8 = 3.$$

Similarly, since

$$4 = 2^2, \quad 2 = 2^1, \quad 1 = 2^0 \quad \text{and} \quad \tfrac{1}{2} = 2^{-1}, \tag{4.3}$$

we have

$$\log_2 4 = 2, \quad \log_2 2 = 1, \quad \log_2 1 = 0 \quad \text{and} \quad \log_2(\tfrac{1}{2}) = -1. \tag{4.4}$$

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ in the 45° line. Figure 4.12 shows the graphs of $y = 2^x$ and $y = \log_2 x$, with the values in equations (4.3) and (4.4) plotted.

Figure 4.12   Graphs of $y = 2^x$ and $y = \log_2 x$

The graph of $y = \log_2 x$ has the y-axis as an asymptote.

Activity 4.4   Evaluating logarithms

(a) Find exact values for each of the following expressions.
  (i) $\log_{10}(10\,000)$ (ii) $\log_3(\tfrac{1}{9})$

(b) Express the meaning of $\log_2 10$ in words, and hence give the exact value of $2^{\log_2 10}$.

Solutions are given on page 56.

Logarithms have a number of useful properties arising from their definition as inverse functions of exponential functions. These are listed below.

Properties of logarithms $(a > 0,\ a \neq 1)$

(a) $\log_a 1 = 0$ and $\log_a a = 1$.

(b) For $x > 0$ and $y > 0$,
  (i) $\log_a(xy) = \log_a x + \log_a y$,
  (ii) $\log_a(x/y) = \log_a x - \log_a y$.

(c) For $x > 0$ and p in $\mathbb{R}$,

$$\log_a(x^p) = p \log_a x.$$

A special case of property (b)(ii) is

$$\log_a(1/y) = -\log_a y.$$

For example, property (b)(i) can be checked as follows.

If we write $p = \log_a x$ and $q = \log_a y$, then $x = a^p$ and $y = a^q$. Thus

$$xy = a^p \times a^q = a^{p+q},$$

See Subsection 3.2.

and hence

$$\log_a(xy) = p + q = \log_a x + \log_a y,$$

as required.

These properties allow us to rearrange many expressions involving logarithms; for example,

$$\log_a\left(\frac{xy}{z}\right)^{1/3} = \tfrac{1}{3}\log_a\left(\frac{xy}{z}\right) \quad \text{(property (c), with } p = \tfrac{1}{3})$$
$$= \tfrac{1}{3}(\log_a(xy) - \log_a z) \quad \text{(property (b)(ii))}$$
$$= \tfrac{1}{3}(\log_a x + \log_a y - \log_a z) \quad \text{(property (b)(i)).}$$

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Activity 4.5   Using the properties of logarithms

Verify each of the following equations.

(a) $\log_a 6 + \log_a 8 - \log_a 2 - \log_a 24 = 0$

(b) $\log_2 \left(\dfrac{x^4 4^{3x}}{2^{x^2}} \right) = 4 \log_2 x + 6x - x^2$

Solutions are given on page 56.

<div style="margin-left: 0;">Base 2 is also used, for example in computer science.</div>

The most commonly used bases for logarithms are 10 and $e = 2.718\,281\ldots$. Logarithms to the base 10 are called **common logarithms** since they were used for many years (from the early 17th Century until calculators were invented) to facilitate multiplication and division. Briefly, the product xy can be evaluated by first finding $\log_{10} x$ and $\log_{10} y$ from logarithm tables, and then finding

$$10^{\log_{10} x + \log_{10} y} = xy,$$

using tables of powers of 10 (called antilogarithms). In this way the problem of *multiplying* x and y is replaced by the simpler problem of *adding* $\log_{10} x$ and $\log_{10} y$. Often, $\log_{10}$ is called simply log.

<div style="margin-left: 0;">The symbol 'ln' is read as 'ell en'. Both the functions log and ln appear on a scientific calculator.</div>

The number $e = 2.718\,281\ldots$ is called the base of **natural logarithms**, and the function $\log_e$ is often called ln. As pointed out in Subsection 3.2, the tangent line to the graph of $y = e^x$ at the point $(0,1)$ has slope 1, and this implies that the tangent line to the graph of $y = \ln x$ at the point $(1,0)$ also has slope 1.

We can use logarithms to solve equations of the form

$$a^x = k,$$

where $k > 0$ and $a > 0$, $a \neq 1$. The solution is $x = \log_a k$, but calculators and computers evaluate logarithms only to base 10 and base e (at the time of writing). To solve an equation such as

$$2^x = 1000, \tag{4.5}$$

<div style="margin-left: 0;">Alternatively, apply $\log_{10}$ to both sides.</div>

we first apply the function ln to both sides:

$$\ln(2^x) = \ln(1000).$$

By property (c), we have

$$x \ln 2 = \ln(1000) \quad \text{and hence} \quad x = \frac{\ln(1000)}{\ln 2} \simeq 9.966.$$

<div style="margin-left: 0;">See Chapter A1.</div>

In particular, we can use logarithms when working with geometric sequences and linear recurrence sequences, to discover how far along the sequence we need to go for the terms to reach a given value. For example, suppose that a savings account contains (in £)

$$s_n = 1000 \times (1.05)^{n-1} \quad (n = 1, 2, 3, \ldots)$$

<div style="margin-left: 0;">See Chapter A1, Activity 3.3.</div>

on 1 January of the nth year, arising from an initial deposit of £1000 and annual interest of 5%. If we leave the money to accumulate at this rate, at the start of which year will the account first contain more than £2000?

To answer this question, we begin by solving

$$1000 \times (1.05)^{n-1} = 2000; \quad \text{that is,} \quad (1.05)^{n-1} = 2.$$

Applying ln to both sides, as we did with equation (4.5), we obtain

$$n - 1 = \frac{\ln 2}{\ln(1.05)} \simeq 14.2, \quad \text{so} \quad n \simeq 15.2.$$

Therefore, at the *start* of the 16th year, the account will (for the first time) contain more than £2000.

Activity 4.6   Trebling the deer population

Suppose that a deer population has size P_n at the start of year n, where

$$P_n = 2666.\dot{6} \times (1.15)^{n-1} + 3333.\dot{3} \quad (n = 1, 2, 3, \ldots).$$

(a) Find the deer population at the start of year 1.

(b) At the start of which year will the population reach more than three times its size at the start of year 1?

Solutions are given on page 56.

Recall that P_n is a *model* of a deer population; see Chapter A1, Exercise 4.3(a).

Summary of Section 4

This section has introduced:

◇   the ideas of one-one, many-one, increasing and decreasing functions;

◇   the fact that a one-one function f has an inverse function f^{-1} which undoes the effect of f and whose domain is the image set of f;

◇   the inverse trigonometric functions arcsine, arccosine and arctangent;

◇   the logarithm functions $\log_a$, where $a > 0$, $a \neq 1$, and their properties;

◇   the use of logarithms to solve equations of the form $a^x = k$.

Exercises for Section 4

Exercise 4.1

For each of the following functions f, find the inverse function f^{-1} and sketch the graph of $y = f^{-1}(x)$.

(a) $f(x) = 4x + 3$ (b) $f(x) = 2x^2 - 24x + 64$ $(x \geq 6)$

For part (b), see the solution to Exercise 2.2(a).

Exercise 4.2

(a) Use your calculator to evaluate each of the following expressions, giving your answers in radians (correct to six significant figures).

   (i) $\arcsin(0.1)$ (ii) $\arccos(-0.85)$ (iii) $\arctan(0.1)$

(b) Write down exact values (in radians) for each of the following expressions.

   (i) $\arcsin\left(-\frac{1}{2}\sqrt{2}\right)$ (ii) $\arccos 1$ (iii) $\arctan(-\sqrt{3})$

Exercise 4.3

(a) Evaluate each of the following expressions, using your calculator where appropriate, giving your answers correct to six significant figures.

(i) $\log_2 64$ (ii) $\log_{10}(0.001)$ (iii) $\ln 1$ (iv) $\ln 10$

(b) Solve the equation $10 = 2^x$ for x, by applying ln to both sides. Hence evaluate $\log_2 10$, correct to six significant figures.

(c) Verify the equation

$$\ln\left(\frac{e^{x+1}}{x^3 + x^2}\right) = x + 1 - 2\ln x - \ln(x + 1).$$

Exercise 4.4

The amount m_n (in £) owing at the start of year n of a mortgage is given by

$$m_n = -6048.6 \times (1.05)^{n-1} + 16\,048.6 \quad (n = 1, 2, \ldots, 20).$$

At the start of which year will the amount owing be less than half the amount owing at the start of year 1?

5 Functions, graphs and equations on the computer

In Section 2, you met the exhibition hall problem, which was solved using the quadratic equation formula. Many problems, however, lead to more complicated equations. For example, a problem about volumes might lead to a *cubic expression*, of the form

$$ax^3 + bx^2 + cx + d \quad \text{(where } a \neq 0\text{)}.$$

More generally, a **polynomial** of **degree** n is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad \text{(where } a_n \neq 0\text{)}.$$

When a polynomial has general degree, as here, we often use sequence notation for the coefficients.

For example, the linear expression $2x + 1$ is a polynomial of degree 1, the quadratic expression $3x^2 - 4$ is a polynomial of degree 2, and the cubic expression $x^3 - x^2 + x - 1$ is a polynomial of degree 3.

A function whose rule involves a polynomial of degree n is called a **polynomial function** of degree n, and an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad \text{(where } a_n \neq 0\text{)}$$

is called a **polynomial equation** of degree n. Polynomial equations of degree 3 are called **cubic equations**.

There is no *simple* general rule for finding the solutions, or roots, of a polynomial equation of degree n greater than 2. Instead, such equations can be solved *approximately* by various methods; for example, we can find approximate solutions by using the computer to plot the graph of the corresponding polynomial function.

In this section the computer is used to plot graphs of functions, and to find approximate solutions of equations using these graphs. You will also meet two non-graphical ways of solving equations.

Refer to Computer Book A for the work in this section.

Summary of Section 5

In this section you saw several methods (including graphical and algebraic methods) for finding solutions of equations, using the computer. You also saw how the computer's graphing capability can be used to find the greatest or least values taken by a function in an interval.

Summary of Chapter A3

In this chapter you met (real) functions, introduced as processors for turning inputs x into outputs y. The graphs of such functions provide a geometric interpretation of them, and when functions are used to solve problems, the shapes of their graphs may provide useful information.

In particular, you saw how to sketch the graphs of quadratic functions, trigonometric functions, exponential functions and various related inverse functions.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Function, domain, rule, image, image set, closed interval, open interval, domain convention, real function, graph of a function, asymptote, modulus, absolute value, quadratic function, completed-square form, horizontal translation, vertical translation, x-scaling, y-scaling, periodic function, sine, cosine and tangent functions, exponential function, one-one, many-one, increasing and decreasing functions, inverse function, arcsine, arccosine and arctangent functions, logarithm, polynomial, cubic equation.

Symbols and notation to know and use

$f(x) = x^2 \ (x \geq 0)$ or $f : x \longmapsto x^2 \ (x \geq 0)$ for a function;
$[a, b]$ and (a, b) for closed and open intervals;
$|x|$ for the modulus of x;
f^{-1} for the inverse function of f;
arcsin, arccos, arctan;
exp, $\log_a$, log, ln.

Mathematical skills

◇ Use function notation and sketch graphs of functions in simple cases.

◇ Use interval notation and the modulus function.

◇ Introduce appropriate functions when solving equations.

◇ Find inverse functions of given one-one functions and sketch their graphs.

◇ Manipulate logarithms.

Modelling skills

◇ Introduce algebraic symbols to represent unknown or general quantities.

◇ Be able to interpret problems concerning areas and volumes in terms of equations, functions and their graphs.

Mathcad skills

◇ Create a function and plot its graph.

◇ Find numerical solutions of equations by using the graph of an appropriate function or by using a solve block.

◇ Solve quadratic equations symbolically.

Ideas to be aware of

◇ Equations may be solved by various techniques, which may be more or less suitable in different cases.

Summary of Block A

This block has introduced several key topics:

◇ linear recurrence sequences;

◇ lines, circles and trigonometry;

◇ functions.

You saw in Chapter A1 that linear recurrence sequences can be used to model various real-life phenomena, such as the height of a bouncing ball and deer populations. These phenomena each involve a list of separate, or discrete, values, so it is appropriate to model them with a sequence which has a subscript, n say, taking integer values. Such models are called **discrete models**, and the subscript n is called a **discrete variable**.

In Chapters A2 and A3, on the other hand, you saw several problems, such as the location of a train or the area of free space in an exhibition hall, which were modelled using formulas or functions involving a variable, x say, taking any value in an interval of the real line. Such models are called **continuous models**, and the variable x is called a **continuous variable**.

When constructing a model, it is usually clear whether a discrete model or a continuous model is appropriate. For example, a discrete model seems suitable for monthly car sales figures, which take separate values, whereas a continuous model seems suitable for the temperature of a cooling cup of tea, which takes a value at every instant of time during a given period. In later blocks of the course, both types of model will be developed further in various ways.

Later in the course, you will meet the word 'continuous' used in a different but related way. A **continuous function** is, roughly speaking, one whose graph can be drawn without lifting the pen from the paper. In this course, most of the functions considered are continuous. For example, all polynomial functions are continuous, and the modulus function, discussed in Section 1, is continuous (but not smooth, since it has a corner at the origin).

This informal definition is made precise, and more general, in courses on pure mathematics.

Solutions to Activities

Solution 1.1

(a) The domain of f is the set of real numbers t satisfying $-1 < t < 2$.

(b) (i) $f(x) = x^2$ $(x \geq 0)$ and $f(1) = 1^2 = 1$.

(ii) $f(x) = 2x + 1$ $(-1 \leq x \leq 1)$ and
$f(1) = 2 \times 1 + 1 = 3$.

Solution 1.2

(a) $(0, 1)$ is open.

(b) $[-3, 2]$ is closed.

(c) $(-2, 2]$ is half-open.

(d) $[0, \infty)$ is closed.

Solution 1.3

(a) Since the square root applies only to a non-negative number, the rule $f(x) = \sqrt{x-1}$ is applicable only to those x for which $x - 1 \geq 0$; that is, $x \geq 1$. Thus the domain of the function f with rule $f(x) = \sqrt{x-1}$ is $[1, \infty)$.

(b) The expression $1/(x-2)$ gives a real number for all x in $\mathbb{R}$ except $x = 2$, and the expression $1/(x+3)$ gives a real number for all x in $\mathbb{R}$ except $x = -3$. Thus the domain of the function with rule $f(x) = 1/(x-2) + 1/(x+3)$ is $\mathbb{R}$ excluding 2 and -3. (This consists of the three open intervals $(-\infty, -3)$, $(-3, 2)$ and $(2, \infty)$.)

Solution 1.4

The ranges of x-values used in this solution are sufficient to indicate the overall shapes of the graphs.

(a)
x	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
x^3	-1	-0.42	-0.13	-0.02	0	0.02	0.13	0.42	1
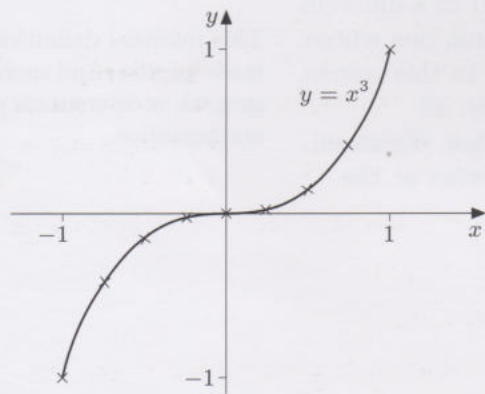
Figure S.1

The image set is the closed interval $[-1, 1]$.

(b)
x	-2	-1.5	-1	-0.5	0.5	1	1.5	2
$1/x^2$	0.25	0.44	1	4	4	1	0.44	0.25
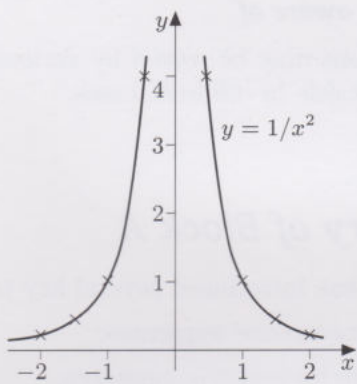
Figure S.2

The image set is the open interval $(0, \infty)$.

Solution 1.5

(a) Since
$$|x|^3 = \begin{cases} x^3, & \text{if } x \geq 0, \\ (-x)^3, & \text{if } x < 0, \end{cases}$$

we can sketch the graph of the function $f(x) = |x|^3$ $(-1 \leq x \leq 1)$ by using the right half of Figure S.1 and its reflection in the y-axis.
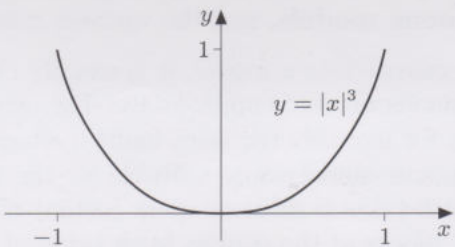
Figure S.3

(b) Since
$$\frac{1}{|x|} = \begin{cases} 1/x, & \text{if } x > 0, \\ 1/(-x), & \text{if } x < 0, \end{cases}$$

we can sketch the graph of $f(x) = 1/|x|$ by using the right half of Figure 1.4 and its reflection in the y-axis.
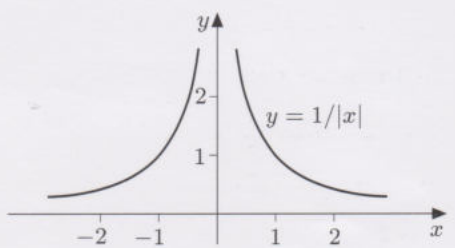
Figure S.4

Solution 2.1

(a) The width of the border can be any number between 0 and $\frac{1}{2} \times 12 = 6$ metres. So the largest closed interval in which x can lie is $[0, 6]$.

(The values $x = 0$ and $x = 6$ are, in a sense, exceptional since $x = 0$ corresponds to 'no border', and $x = 6$ corresponds to 'no clear space'. Such exceptional points are often included in the domain of a function which is part of a model, provided that the rule of the function is applicable at the points.)

(b) The length of the clear space is $16 - 2x$, and the width is $12 - 2x$.

(c) We have
$$A = (16 - 2x)(12 - 2x)$$
$$= 192 - 56x + 4x^2.$$

Solution 2.2

(a) The equation
$$4x^2 - 56x + 96 = 0$$
is equivalent to
$$x^2 - 14x + 24 = 0.$$

The solutions are therefore
$$x = \frac{14 \pm \sqrt{(-14)^2 - 4 \times 24}}{2}$$
$$= \frac{1}{2}(14 \pm \sqrt{100});$$
that is,
$$x = 12 \quad \text{and} \quad x = 2.$$

These solutions could also be found by factorising the expression $x^2 - 14x + 24$.

(b) Of these solutions, only $x = 2$ lies in the interval $[0, 6]$, so this gives the solution to the exhibition hall problem: the border width is 2 metres.

Solution 2.3

(a) Equation (2.2) can be written as
$$4x^2 - 56x + 192 = 144;$$
that is,
$$4x^2 - 56x + 48 = 0,$$
which is equivalent to
$$x^2 - 14x + 12 = 0.$$

The solutions are therefore
$$x = \frac{14 \pm \sqrt{(-14)^2 - 4 \times 12}}{2}$$
$$= \frac{1}{2}(14 \pm \sqrt{148})$$
$$= 7 \pm \sqrt{37};$$
that is,
$$x \simeq 13.083 \quad \text{and} \quad x \simeq 0.917.$$

(b) Of these solutions, only $x \simeq 0.917$ lies in the interval $[0, 6]$, so this gives the solution to the modified exhibition hall problem. The width of the border is 0.917 metres.

(As predicted using the graph, the solution is approximately 1.)

Solution 2.4

(a) Using equation (2.3) with $p = 2$, we obtain
$$x^2 + 4x + 3 = (x + 2)^2 - 4 + 3$$
$$= (x + 2)^2 - 1.$$

(b) Using equation (2.3) with $p = \frac{1}{2}$, we obtain
$$x^2 + x + \frac{3}{4} = (x + \frac{1}{2})^2 - \frac{1}{4} + \frac{3}{4}$$
$$= (x + \frac{1}{2})^2 + \frac{1}{2}.$$

(c) First take out the factor 4, to give
$$4x^2 - 56x + 192 = 4(x^2 - 14x + 48),$$
and then use equation (2.3) with $p = -7$:
$$4x^2 - 56x + 192 = 4((x - 7)^2 - 49 + 48)$$
$$= 4((x - 7)^2 - 1)$$
$$= 4(x - 7)^2 - 4.$$

(d) First take out the factor $\frac{1}{2}$, to give
$$\frac{1}{2}x^2 + x = \frac{1}{2}(x^2 + 2x),$$
and then use equation (2.3) with $p = 1$:
$$\frac{1}{2}x^2 + x = \frac{1}{2}((x + 1)^2 - 1)$$
$$= \frac{1}{2}(x + 1)^2 - \frac{1}{2}.$$

Solution 2.5

By the solution to Activity 2.4(c),
$$f(x) = 4x^2 - 56x + 192$$
$$= 4(x - 7)^2 - 4,$$
so the graph of $y = f(x)$ can be obtained from the graph of $y = x^2$ by performing:

◇ a y-scaling with factor 4;

◇ a horizontal translation by 7 units to the right;

◇ a vertical translation by 4 units downwards.

(The order of performing the y-scaling and the horizontal translation may be reversed.)

The vertex of the parabola $y = f(x)$ is at $(7, -4)$.

The y-intercept is
$$f(0) = 192,$$
and the x-intercepts are $x = 6$ and $x = 8$. The x-intercepts are found by solving the equation
$$f(x) = 4x^2 - 56x + 192 = 0,$$
which is equivalent to
$$x^2 - 14x + 48 = (x - 6)(x - 8) = 0.$$

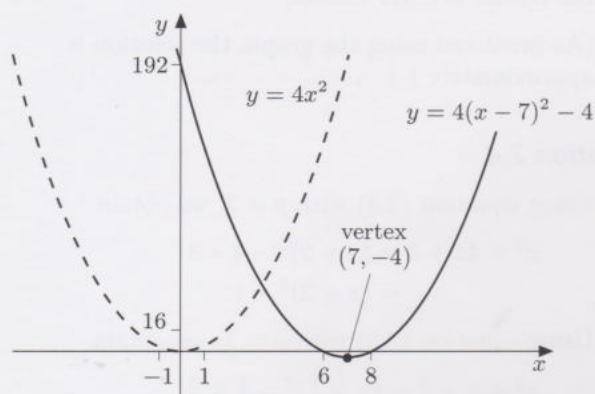The stages in the process, other than the y-scaling, are shown in Figure S.5.

Figure S.5

Solution 3.1

As the point (x, y) rotates around the unit circle, each of the x- and y-coordinates varies between -1 and 1 (both inclusive). Hence the complete set of image values for $\cos t = x$ and $\sin t = y$ is, in each case, the interval $[-1, 1]$.

Solution 3.2

(a) For the inputs 0 and $\frac{1}{2}\pi$, it is easiest to refer directly to the unit circle. If $t = 0$, then the corresponding point on the unit circle is $(x, y) = (1, 0)$, so

$$\cos 0 = 1, \quad \sin 0 = 0.$$

If $t = \frac{1}{2}\pi$, then $(x, y) = (0, 1)$, so

$$\cos(\tfrac{1}{2}\pi) = 0, \quad \sin(\tfrac{1}{2}\pi) = 1.$$

For the inputs $\frac{1}{6}\pi$ and $\frac{1}{3}\pi$, it is easiest to refer to the following right-angled triangle, and to use the expressions for cos and sin in terms of the opposite and adjacent sides and the hypotenuse of the triangle (Chapter A2, Subsection 3.2).
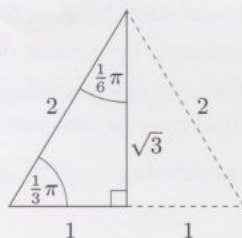
Figure S.6

This gives

$$\cos(\tfrac{1}{6}\pi) = \tfrac{1}{2}\sqrt{3} \simeq 0.87, \quad \sin(\tfrac{1}{6}\pi) = \tfrac{1}{2} = 0.5,$$
$$\cos(\tfrac{1}{3}\pi) = \tfrac{1}{2} = 0.5, \quad \sin(\tfrac{1}{3}\pi) = \tfrac{1}{2}\sqrt{3} \simeq 0.87.$$

For the input $\frac{1}{4}\pi$, we use the following right-angled triangle in a similar way.
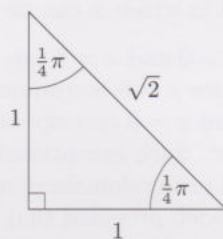
Figure S.7

This gives

$$\cos(\tfrac{1}{4}\pi) = \sin(\tfrac{1}{4}\pi) = 1/\sqrt{2} = \tfrac{1}{2}\sqrt{2} \simeq 0.71.$$

(b) Use of the formulas indicated gives

$$\cos(\tfrac{2}{3}\pi) = -\cos(\tfrac{1}{3}\pi) = -\tfrac{1}{2} = -0.5,$$
$$\sin(\tfrac{2}{3}\pi) = \sin(\tfrac{1}{3}\pi) = \tfrac{1}{2}\sqrt{3} \simeq 0.87,$$
$$\cos(\tfrac{3}{4}\pi) = -\cos(\tfrac{1}{4}\pi) = -\tfrac{1}{2}\sqrt{2} \simeq -0.71,$$
$$\sin(\tfrac{3}{4}\pi) = \sin(\tfrac{1}{4}\pi) = \tfrac{1}{2}\sqrt{2} \simeq 0.71,$$
$$\cos(\tfrac{5}{6}\pi) = -\cos(\tfrac{1}{6}\pi) = -\tfrac{1}{2}\sqrt{3} \simeq -0.87,$$
$$\sin(\tfrac{5}{6}\pi) = \sin(\tfrac{1}{6}\pi) = \tfrac{1}{2} = 0.5,$$
$$\cos \pi = -\cos 0 = -1,$$
$$\sin \pi = \sin 0 = 0.$$

(c) Use of the formulas indicated gives

$$\cos(-\tfrac{1}{6}\pi) = \cos(\tfrac{1}{6}\pi) = \tfrac{1}{2}\sqrt{3} \simeq 0.87,$$
$$\sin(-\tfrac{1}{6}\pi) = -\sin(\tfrac{1}{6}\pi) = -\tfrac{1}{2} = -0.5,$$
$$\cos(-\tfrac{1}{4}\pi) = \cos(\tfrac{1}{4}\pi) = \tfrac{1}{2}\sqrt{2} \simeq 0.71,$$
$$\sin(-\tfrac{1}{4}\pi) = -\sin(\tfrac{1}{4}\pi) = -\tfrac{1}{2}\sqrt{2} \simeq -0.71,$$
$$\cos(-\tfrac{1}{3}\pi) = \cos(\tfrac{1}{3}\pi) = \tfrac{1}{2} = 0.5,$$
$$\sin(-\tfrac{1}{3}\pi) = -\sin(\tfrac{1}{3}\pi) = -\tfrac{1}{2}\sqrt{3} \simeq -0.87,$$
$$\cos(-\tfrac{1}{2}\pi) = \cos(\tfrac{1}{2}\pi) = 0,$$
$$\sin(-\tfrac{1}{2}\pi) = -\sin(\tfrac{1}{2}\pi) = -1,$$
$$\cos(-\tfrac{2}{3}\pi) = \cos(\tfrac{2}{3}\pi) = -\tfrac{1}{2} = -0.5,$$
$$\sin(-\tfrac{2}{3}\pi) = -\sin(\tfrac{2}{3}\pi) = -\tfrac{1}{2}\sqrt{3} \simeq -0.87,$$
$$\cos(-\tfrac{3}{4}\pi) = \cos(\tfrac{3}{4}\pi) = -\tfrac{1}{2}\sqrt{2} \simeq -0.71,$$
$$\sin(-\tfrac{3}{4}\pi) = -\sin(\tfrac{3}{4}\pi) = -\tfrac{1}{2}\sqrt{2} \simeq -0.71,$$
$$\cos(-\tfrac{5}{6}\pi) = \cos(\tfrac{5}{6}\pi) = -\tfrac{1}{2}\sqrt{3} \simeq -0.87,$$
$$\sin(-\tfrac{5}{6}\pi) = -\sin(\tfrac{5}{6}\pi) = -\tfrac{1}{2} = -0.5,$$
$$\cos(-\pi) = \cos \pi = -1,$$
$$\sin(-\pi) = -\sin \pi = 0.$$

Solution 3.3

(a) The graph of $y = -3 + \cos x$ is obtained from that of $y = \cos x$ by translating downwards by 3 units. This gives the graph in Figure S.8.
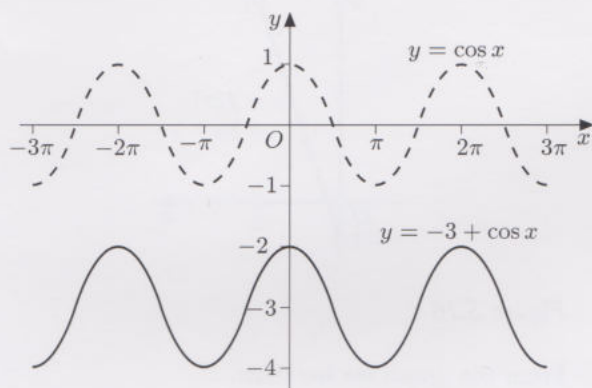

Figure S.8

(b) The graph of $y = \frac{1}{2}\cos x$ is obtained from that of $y = \cos x$ by performing a y-scaling with factor $\frac{1}{2}$. This gives the graph in Figure S.9.
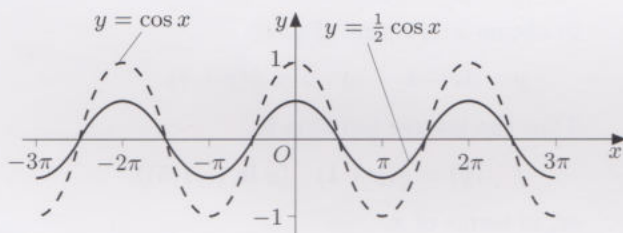

Figure S.9
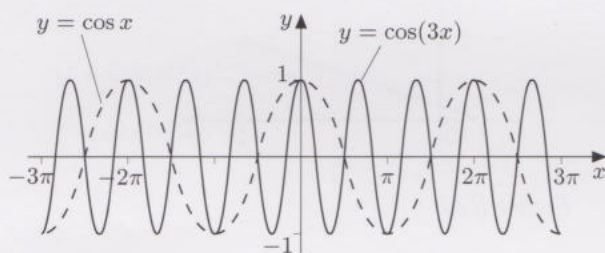
(c) The graph of $y = \cos(3x)$ is obtained from that of $y = \cos x$ by performing an x-scaling with factor $\frac{1}{3}$. This gives the graph in Figure S.10.

Figure S.10

(d) The graph of $y = \cos(-x)$ is obtained from that of $y = \cos x$ by performing an x-scaling with factor -1. This is the same as reflecting the graph in the y-axis, which gives the graph of $y = \cos x$ once more. This corresponds to the fact that $\cos(-x) = \cos x$ for all x.

Solution 3.4

(a) From the definition of tan, we have

$$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0,$$

$$\tan(\tfrac{1}{6}\pi) = \frac{\sin(\tfrac{1}{6}\pi)}{\cos(\tfrac{1}{6}\pi)} = \frac{\tfrac{1}{2}}{\tfrac{1}{2}\sqrt{3}} = \tfrac{1}{3}\sqrt{3} \simeq 0.58,$$

$$\tan(\tfrac{1}{4}\pi) = \frac{\sin(\tfrac{1}{4}\pi)}{\cos(\tfrac{1}{4}\pi)} = \frac{\tfrac{1}{2}\sqrt{2}}{\tfrac{1}{2}\sqrt{2}} = 1,$$

$$\tan(\tfrac{1}{3}\pi) = \frac{\sin(\tfrac{1}{3}\pi)}{\cos(\tfrac{1}{3}\pi)} = \frac{\tfrac{1}{2}\sqrt{3}}{\tfrac{1}{2}} = \sqrt{3} \simeq 1.73.$$

(b) Use of the formula $\tan(-x) = -\tan x$ gives

$$\tan(-\tfrac{1}{6}\pi) = -\tan(\tfrac{1}{6}\pi) = -\tfrac{1}{3}\sqrt{3} \simeq -0.58,$$

$$\tan(-\tfrac{1}{4}\pi) = -\tan(\tfrac{1}{4}\pi) = -1,$$

$$\tan(-\tfrac{1}{3}\pi) = -\tan(\tfrac{1}{3}\pi) = -\sqrt{3} \simeq -1.73.$$

Solution 3.5

(a) The required values (correct to two decimal places) of 5^x are shown in the following table.

x	-1	-0.75	-0.5	-0.25	0
5^x	0.2	0.30	0.45	0.67	1

x	0	0.25	0.5	0.75	1
5^x	1	1.50	2.24	3.34	5

(b) The graph is shown in Figure S.11.
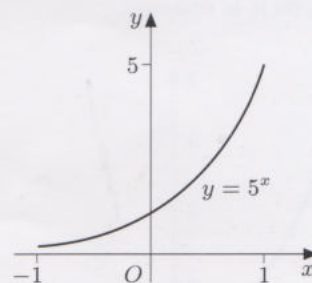
Figure S.11

(c) We have $g(x) = (\tfrac{1}{5})^x = 5^{-x}$. The graph of this function is obtained from that of $f(x) = 5^x$ by reflection in the y-axis, which is equivalent to an x-scaling with factor -1. Hence the graph of $g(x) = (\tfrac{1}{5})^x$ is as in Figure S.12.
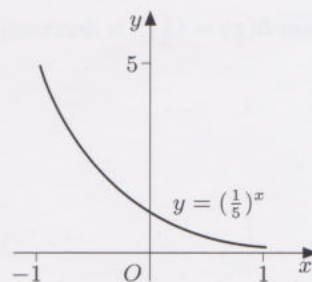
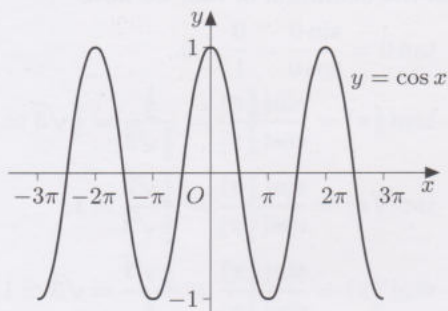*Figure S.12*

Solution 4.1

(a)

Figure S.13

The function $f(x) = \cos x$ is neither increasing nor decreasing, and it is many-one.
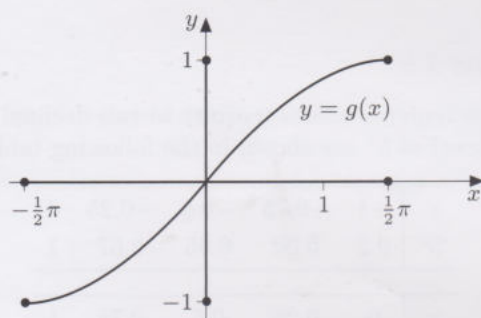
(b)

Figure S.14

The function $g(x) = \sin x$ $(-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi)$ is increasing, so it is one-one.
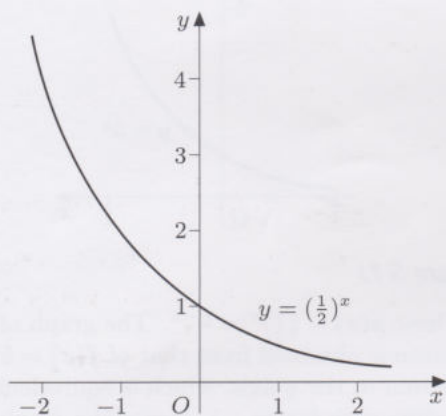
(c)

Figure S.15

The function $h(x) = (\frac{1}{2})^x$ is decreasing, so it is one-one.
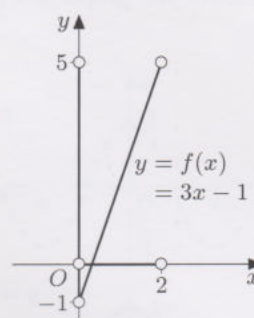
Solution 4.2

(a) The graph of f is as in Figure S.16.

Figure S.16

From the graph we see that:

◇   the function f is increasing, and so one-one;

◇   the image set of f is $(-1, 5)$.

Therefore f has an inverse function f^{-1} with domain $(-1, 5)$ and image set $(0, 2)$.

We can find the rule of f^{-1} by solving

$$y = f(x) = 3x - 1$$

to obtain x in terms of y:

$$y = 3x - 1, \quad \text{so} \quad x = \tfrac{1}{3}(y + 1).$$

Thus the inverse function is

$$f^{-1}(y) = \tfrac{1}{3}(y + 1) \quad (y \text{ in } (-1, 5)),$$

or, in terms of x,

$$f^{-1}(x) = \tfrac{1}{3}(x + 1) \quad (x \text{ in } (-1, 5)).$$

The graph of $y = f^{-1}(x)$ is found by reflecting that of $y = f(x)$ (Figure S.16) in the $45°$ line, and is shown in Figure S.17.
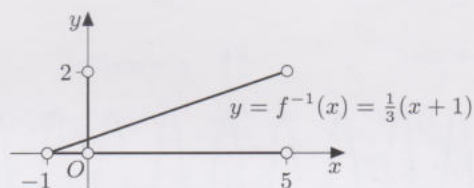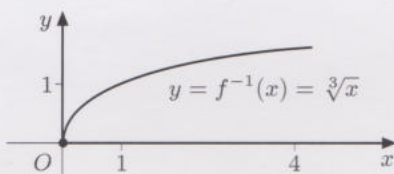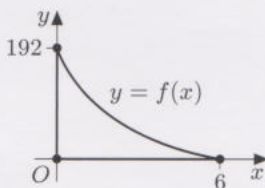

Figure S.17

(b) The graph of f is as in Figure S.18.
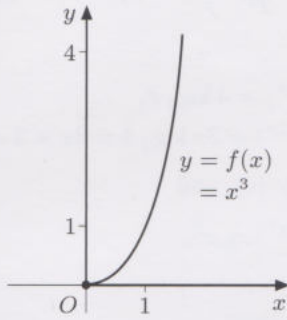
Figure S.18

From the graph we see that:

◇   the function f is increasing, and so one-one;

◇   the image set of f is $[0, \infty)$.

Therefore f has an inverse function f^{-1} with domain $[0, \infty)$ and image set $[0, \infty)$.

We can find the rule of f^{-1} by solving

$$y = f(x) = x^3, \quad \text{where} \quad y \geq 0,$$

to obtain x in terms of y:

$$y = x^3, \quad \text{so} \quad x = \sqrt[3]{y}.$$

Thus the inverse function is

$$f^{-1}(y) = \sqrt[3]{y} \quad (y \text{ in } [0, \infty)),$$

or, in terms of x,

$$f^{-1}(x) = \sqrt[3]{x} \quad (x \text{ in } [0, \infty)).$$

The graph of $y = f^{-1}(x)$ is found by reflecting that of $y = f(x)$ (Figure S.18) in the $45°$ line, and is shown in Figure S.19.

Figure S.19

(c) The graph of f is as in Figure S.20; it can be obtained from Figure S.5 by restricting the domain to $[0, 6]$.

Figure S.20

From the graph we see that:

◇   the function f is decreasing, and so one-one;

◇   the image set of f is $[0, 192]$.

Therefore f has an inverse function f^{-1} with domain $[0, 192]$ and image set $[0, 6]$.

We can find the rule of f^{-1} by solving

$$y = f(x) = 4x^2 - 56x + 192,$$
$$\text{where} \quad 0 \leq y \leq 192,$$

to obtain x in terms of y:

$$4x^2 - 56x + 192 - y = 0,$$

so

$$x = \frac{56 \pm \sqrt{(-56)^2 - 16(192 - y)}}{8}$$

$$= \tfrac{1}{8}\left(56 \pm \sqrt{64 + 16y}\right)$$

$$= 7 \pm \sqrt{1 + \tfrac{1}{4}y}.$$

(Alternatively, on completing the square we obtain

$$y = 4x^2 - 56x + 192 = 4(x - 7)^2 - 4,$$

which can be rearranged as

$$(x - 7)^2 = 1 + \tfrac{1}{4}y,$$

giving $x = 7 \pm \sqrt{1 + \tfrac{1}{4}y}$, as before.)

Now, we need to choose the solution of this equation which lies in $[0, 6]$, the domain of f. Therefore we choose

$$x = 7 - \sqrt{1 + \tfrac{1}{4}y}.$$

Thus the inverse function is

$$f^{-1}(y) = 7 - \sqrt{1 + \tfrac{1}{4}y} \quad (y \text{ in } [0, 192]),$$

or, in terms of x,

$$f^{-1}(x) = 7 - \sqrt{1 + \tfrac{1}{4}x} \quad (x \text{ in } [0, 192]).$$

The graph of $y = f^{-1}(x)$ is found by reflecting that of $y = f(x)$ (Figure S.20) in the $45°$ line, and is shown in Figure S.21.
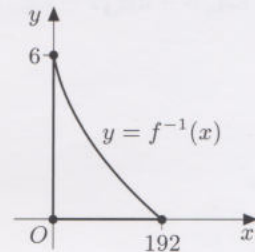

Figure S.21

Solution 4.3

(a) (i) The domain of arcsin is $[-1, 1]$. Since 2π does not lie in this interval, $\arcsin(2\pi)$ is not valid.

(ii) The domain of arccos is $[-1, 1]$. Since 0 lies in this interval, $\arccos 0$ is valid.

(iii) and (iv) The domain of arctan is $\mathbb{R}$. Since $\frac{1}{2}\pi$ and $\tan(\frac{5}{4}\pi)$ are real numbers, $\arctan(\frac{1}{2}\pi)$ and $\arctan(\tan(\frac{5}{4}\pi))$ are both valid.

(b) (i) $\arcsin(\frac{1}{2}\sqrt{3}) = \frac{1}{3}\pi$

(ii) $\arccos(-0.5) = \frac{2}{3}\pi$

(iii) $\arctan 1 = \frac{1}{4}\pi$

(iv) $\arcsin(\sin(\frac{1}{5}\pi)) = \frac{1}{5}\pi$

(v) $\cos(\arccos(0.9)) = 0.9$

(vi) The value $\frac{5}{4}\pi$ does not lie in the domain of the function

$$f(x) = \tan x \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi),$$

but $\frac{1}{4}\pi$ does lie in this domain, and $\tan(\frac{5}{4}\pi) = \tan(\frac{1}{4}\pi)$. So

$$\arctan(\tan(\tfrac{5}{4}\pi)) = \arctan(\tan(\tfrac{1}{4}\pi)) = \tfrac{1}{4}\pi.$$

Solution 4.4

(a) (i) $\log_{10}(10\,000) = \log_{10}(10^4) = 4$

(ii) $\log_3(\frac{1}{9}) = \log_3(3^{-2}) = -2$

(b) Since $\log_2 10$ is the power to which 2 must be raised to obtain 10,

$$2^{\log_2 10} = 10.$$

Solution 4.5

(a) By properties (b)(i) and (b)(ii),

$$\log_a 6 + \log_a 8 - \log_a 2 - \log_a 24$$
$$= \log_a\left(\frac{6 \times 8}{2 \times 24}\right) = \log_a 1.$$

Hence, by property (a),

$$\log_a 6 + \log_a 8 - \log_a 2 - \log_a 24 = 0.$$

(b) By properties (b)(i) and (b)(ii),

$$\log_2\left(\frac{x^4 4^{3x}}{2^{x^2}}\right) = \log_2(x^4) + \log_2(4^{3x}) - \log_2(2^{x^2}).$$

Now

$$\log_2(x^4) = 4\log_2 x,$$
$$\log_2(4^{3x}) = 3x\log_2 4 = 3x \times 2 = 6x,$$

by property (c), and

$$\log_2(2^{x^2}) = x^2,$$

so

$$\log_2\left(\frac{x^4 4^{3x}}{2^{x^2}}\right) = 4\log_2 x + 6x - x^2.$$

Solution 4.6

(a) At the start of year 1, the deer population is

$$P_1 = 2666.\dot{6} \times 1 + 3333.\dot{3} = 6000.$$

(b) We need to solve the equation

$$P_n = 3 \times 6000 = 18\,000;$$

that is,

$$2666.\dot{6} \times (1.15)^{n-1} + 3333.\dot{3} = 18\,000,$$

or

$$(1.15)^{n-1} = \frac{18\,000 - 3333.\dot{3}}{2666.\dot{6}} = 5.5.$$

Applying ln to both sides and rearranging, we obtain

$$n - 1 = \frac{\ln(5.5)}{\ln(1.15)} \simeq 12.2, \quad \text{so} \quad n \simeq 13.2.$$

So the population will exceed $18\,000$ at the start of year 14.

Solutions to Exercises

Solution 1.1

(a) $(-\infty, 1)$

(b) The expression $\sqrt{x}$ is defined for $x \geq 0$, so $1/\sqrt{x}$ is defined for $x > 0$. Thus, by the domain convention, the domain is $(0, \infty)$.

(c) The expression $\sqrt{9+x}$ is defined for $9 + x \geq 0$, which is equivalent to $x \geq -9$. Thus, by the domain convention, the domain is $[-9, \infty)$.

Solution 1.2

(a)

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
x^4	16	5.06	1	0.06	0	0.06	1	5.06	16
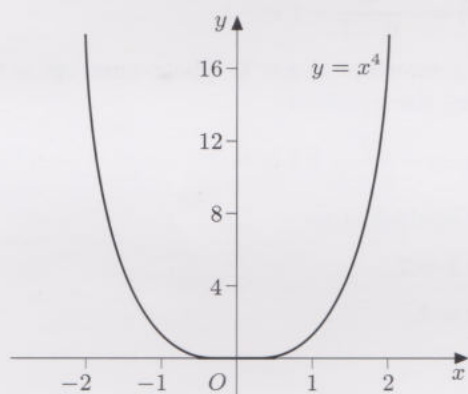
Figure S.22

The image set is the closed interval $[0, \infty)$.

(b)

x	0.1	0.5	1	1.5	2
$1/\sqrt{x}$	3.16	1.41	1	0.82	0.71
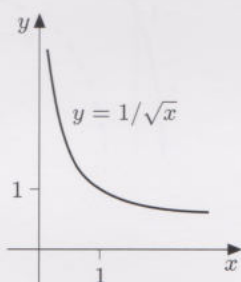
Figure S.23

The image set is the open interval $(0, \infty)$.

(c) Since this function has the property that

$$f(-x) = \frac{1}{\sqrt{|-x|}} = \frac{1}{\sqrt{|x|}} = f(x),$$

it is symmetric in the y-axis. Now

$$\frac{1}{\sqrt{|x|}} = \frac{1}{\sqrt{x}} \quad \text{for } x > 0,$$

so we can use the solution to part (b), together with its reflection in the y-axis.
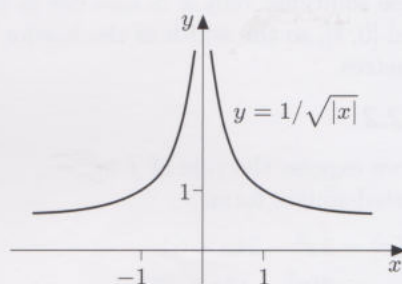
Figure S.24

The image set is the open interval $(0, \infty)$.

Solution 2.1

(a) The largest closed interval in which x can lie is $[0, 4]$.
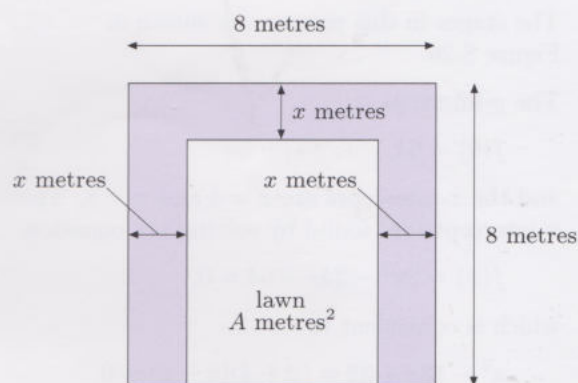
Figure S.25

(b) The lawn has length $8 - x$ and width $8 - 2x$, so its area is

$$A = (8 - x)(8 - 2x)$$
$$= 64 - 24x + 2x^2.$$

(c) Half the area of the garden is 32, so x must satisfy

$$64 - 24x + 2x^2 = 32;$$

that is,

$$2x^2 - 24x + 32 = 0,$$

or, equivalently,

$$x^2 - 12x + 16 = 0.$$

The solutions of this equation are

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4 \times 16}}{2}$$
$$= \tfrac{1}{2}(12 \pm \sqrt{80})$$
$$= 6 \pm 2\sqrt{5};$$

that is,

$$x \simeq 10.47 \quad \text{and} \quad x \simeq 1.53.$$

Of these solutions, only $x \simeq 1.53$ lies in the interval $[0, 4]$, so the width of the border is 1.53 metres.

Solution 2.2

(a) First, we express the rule of f in completed-square form:

$$f(x) = 2x^2 - 24x + 64$$
$$= 2(x^2 - 12x + 32)$$
$$= 2((x - 6)^2 - 36 + 32)$$
$$= 2(x - 6)^2 - 8.$$

Therefore the graph of $y = f(x)$ can be obtained from the graph of $y = x^2$ by performing:

◇   a y-scaling with factor 2;

◇   a horizontal translation by 6 units to the right;

◇   a vertical translation by 8 units downwards.

The stages in this process are shown in Figure S.26.

The y-intercept is

$$f(0) = 64,$$

and the x-intercepts are $x = 4$ and $x = 8$. The x-intercepts are found by solving the equation

$$f(x) = 2x^2 - 24x + 64 = 0,$$

which is equivalent to

$$x^2 - 12x + 32 = (x - 4)(x - 8) = 0.$$
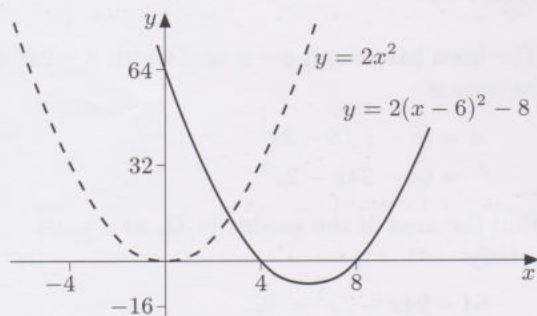
Figure S.26

(b) In the garden problem, we needed to find a solution of the equation

$$f(x) = 2x^2 - 24x + 64 = 32$$

which lies in the interval $[0, 4]$. The graph plotted in part (a) shows that the solution of this equation which lies in the interval $[0, 4]$ is between 1 and 2, as found in the solution to Exercise 2.1(c).

Solution 2.3

The graph of $y = f(x)$ can be obtained from the graph of $y = 1/x$ (see Figure 1.4) by performing:

◇   a y-scaling with factor -2;

◇   a horizontal translation by 1 unit to the left;

◇   a vertical translation by 1 unit upwards.

The stages in this process are shown in Figure S.27.

The y-intercept is

$$f(0) = -\frac{2}{0 + 1} + 1 = -1,$$

and the x-intercept is $x = 1$. The x-intercept is found by solving the equation

$$f(x) = -\frac{2}{x + 1} + 1 = 0,$$

which is equivalent to
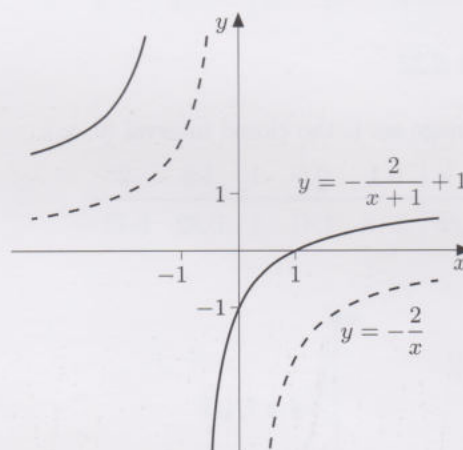
$$x + 1 = 2,$$

giving $x = 1$.

Figure S.27

Solution 3.1

(a) The graph of $y = 1 + \sin x$ is obtained from that of $y = \sin x$ by translating upwards by 1 unit. This gives the graph in Figure S.28.
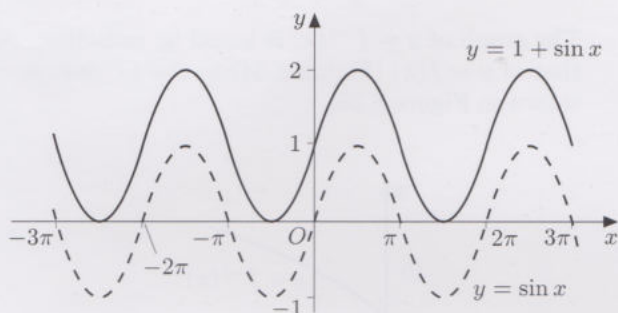

Figure S.28

(b) The graph of $y = -\sin x$ is obtained from that of $y = \sin x$ by performing a y-scaling with factor -1. This is the same as reflection in the x-axis, and gives the graph in Figure S.29.
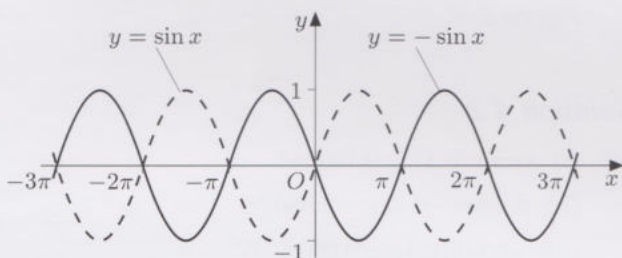

Figure S.29

(c) The graph of $y = \sin(4x)$ is obtained from that of $y = \sin x$ by performing an x-scaling with factor $\frac{1}{4}$. This gives the graph in Figure S.30.
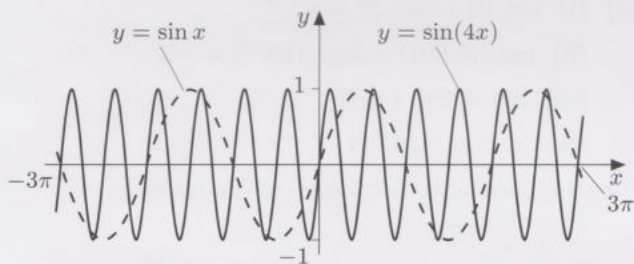
Figure S.30

Solution 3.2

The graph of $g(x) = e^{x+1}$ is obtained from that of $f(x) = e^x$ by applying a horizontal translation by 1 unit to the left. This gives the graph in Figure S.31.
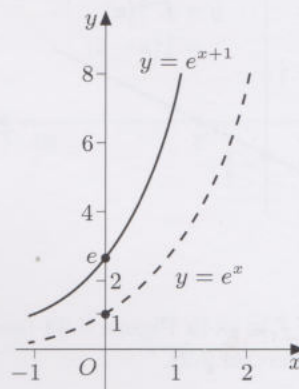
Figure S.31

(Note that the graph of g may also be obtained from that of f by applying a y-scaling with factor e, since $e^{x+1} = e \times e^x$.)

Solution 4.1

(a) The graph of f is as in Figure S.32.
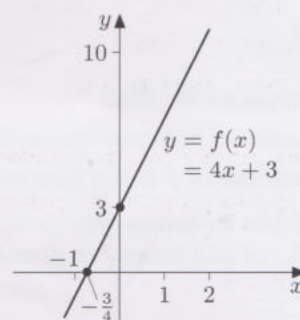
Figure S.32

From the graph we see that:

◇ the function f is increasing, and so one-one;

◇ the image set of f is the whole of $\mathbb{R}$.

Therefore f has an inverse function f^{-1} with domain $\mathbb{R}$ and image set $\mathbb{R}$, the domain of f.

We can find the rule of f^{-1} by solving

$$y = f(x) = 4x + 3$$

to obtain x in terms of y:

$$y = 4x + 3, \quad \text{so} \quad x = \tfrac{1}{4}(y - 3).$$

Thus the inverse function is

$$f^{-1}(y) = \tfrac{1}{4}(y - 3),$$

or, in terms of x,

$$f^{-1}(x) = \tfrac{1}{4}(x - 3).$$

The graph of $y = f^{-1}(x)$ is found by reflecting that of $y = f(x)$ (Figure S.32) in the 45° line, as shown in Figure S.33.
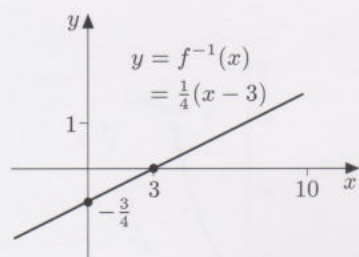

Figure S.33

(b) The graph of f is as in Figure S.34 (see the solution to Exercise 2.2).
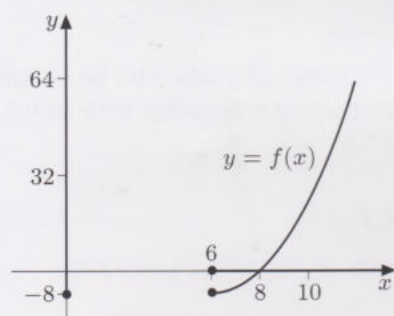

Figure S.34

From the graph we see that:

◇ the function f is increasing, and so one-one;

◇ the image set of f is the interval $[-8, \infty)$.

Therefore f has an inverse function f^{-1} with domain $[-8, \infty)$ and image set $[6, \infty)$, the domain of f.

We can find the rule of f^{-1} by solving

$$y = f(x) = 2x^2 - 24x + 64, \text{ where } y \geq -8,$$

to obtain x in terms of y:

$$2x^2 - 24x + 64 - y = 0,$$

so

$$x = \frac{24 \pm \sqrt{24^2 - 8(64 - y)}}{4}$$
$$= \tfrac{1}{4}\left(24 \pm \sqrt{64 + 8y}\right)$$
$$= 6 \pm \sqrt{4 + \tfrac{1}{2}y}.$$

Now, we need to choose the solution of this equation which lies in $[6, \infty)$, the image set of f^{-1}. Therefore we choose

$$x = 6 + \sqrt{4 + \tfrac{1}{2}y}.$$

Thus the inverse function is

$$f^{-1}(y) = 6 + \sqrt{4 + \tfrac{1}{2}y} \quad (y \text{ in } [-8, \infty)),$$

or, in terms of x,

$$f^{-1}(x) = 6 + \sqrt{4 + \tfrac{1}{2}x} \quad (x \text{ in } [-8, \infty)).$$

The graph of $y = f^{-1}(x)$ is found by reflecting that of $y = f(x)$ (Figure S.34) in the 45° line, as shown in Figure S.35.
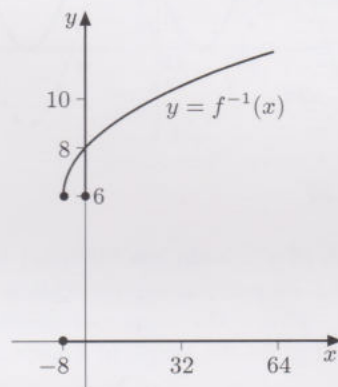

Figure S.35

Solution 4.2

(a) (i) $\arcsin(0.1) \simeq 0.100\,167$

    (ii) $\arccos(-0.85) \simeq 2.586\,78$

    (iii) $\arctan(0.1) \simeq 0.099\,668\,7$

(b) (i) $\arcsin(-\tfrac{1}{2}\sqrt{2}) = -\tfrac{1}{4}\pi$

    (ii) $\arccos 1 = 0$

    (iii) $\arctan(-\sqrt{3}) = -\tfrac{1}{3}\pi$

Solution 4.3

(a) (i) $\log_2 64 = \log_2 2^6 = 6$

    (ii) $\log_{10}(0.001) = \log_{10}(10^{-3}) = -3$

    (iii) $\ln 1 = \log_e 1 = 0$

    (iv) $\ln 10 \simeq 2.302\,59$

(b) Applying ln to both sides of the equation $10 = 2^x$ gives

$$\ln(2^x) = \ln 10, \quad \text{so} \quad x \ln 2 = \ln 10,$$

giving

$$x = \frac{\ln 10}{\ln 2} \simeq 3.321\,93.$$

Since $10 = 2^x$,

$$\log_2 10 = \log_2(2^x)$$
$$= x \simeq 3.321\,93.$$

(c) By properties (b)(i) and (ii),

$$\ln\left(\frac{e^{x+1}}{x^3 + x^2}\right) = \ln(e^{x+1}) - \ln(x^3 + x^2)$$

$$= x + 1 - \ln(x^2(x+1))$$
$$= x + 1 - \ln(x^2) - \ln(x+1)$$
$$= x + 1 - 2\ln x - \ln(x+1),$$

as required.

Solution 4.4

At the start of year 1, the amount owing is

$$m_1 = -6048.6 \times (1.05)^0 + 16\,048.6$$
$$= 10\,000.$$

Thus we need to solve the equation

$$m_n = \tfrac{1}{2} \times 10\,000 = 5000;$$

that is,

$$-6048.6 \times (1.05)^{n-1} + 16\,048.6 = 5000,$$

or

$$(1.05)^{n-1} = \frac{5000 - 16\,048.6}{-6048.6} \simeq 1.826\,637\,6.$$

Applying ln to both sides and rearranging, we obtain

$$n - 1 = \frac{\ln(1.826\,637\,6)}{\ln(1.05)} \simeq 12.3, \quad \text{so} \quad n \simeq 13.3.$$

So the amount owing will be less than £5000 at the start of year 14.

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